A UNIVALENCY CRITERION

J. M. Anderson and A. Hinkkanen

Dedicated to George Piranian

1. Introduction. Let f be a meromorphic and locally univalent function in the upper half-plane U, that is, $f'(z) \neq 0$ and any pole of f is simple. It is natural, when looking for criteria which imply the univalence of f, to introduce the Schwarzian derivative $S(f,z)$, defined by

$$S(f,z) = \left(\frac{f''}{f'}\right)'(z) - \frac{1}{2}\left(\frac{f''}{f'}\right)^2(z).$$

We shall use the notation

$$U = \{z : \operatorname{Im} z > 0\}, \qquad L = \{z : \operatorname{Im} z < 0\}, \qquad B(z,r) = \{w : |w-z| \leq r\}.$$

If f can be extended to a local homeomorphism F defined on the whole sphere $\bar{\mathbf{C}}$ then f will be univalent in U. This method for establishing univalence was emphasized by Ahlfors in [1], where he gave extensions and alternative derivations of many known criteria for univalence. If F is differentiable at $z = z_0$, say, the condition $|F_{\bar{z}}| < |F_z|$ for $z = z_0$ ensures that the Jacobian of F is not zero at z_0 and hence that F is a local homeomorphism at z_0. The stronger condition $|F_{\bar{z}}| \leq k|F_z|$ for all $z \in L$, where $0 < k < 1$, says that f has a k-quasiconformal extension to L. This is not the standard terminology, but agrees with that used by Ahlfors in [1]. Thus for $0 < k < 1$, a k-quasiconformal mapping is one whose maximal dilatation does not exceed $(1+k)/(1-k)$. Ahlfors has proved the following result [1, p. 29].

THEOREM A. *Suppose that* $0 < k < 1$, $|c-1| \leq k$ *and* $y = \operatorname{Im} z$. *If f is meromorphic and locally univalent in U and such that*

$$\left| 2y^2 S(f,z) - c(c-1)\left(\frac{\bar{z}+it}{z+it}\right)^2 \right| \leq k|c| \tag{1.1}$$

for all $z \in U$ *and some* $t > 0$, *then f is univalent in U and has a k-quasiconformal extension to* $\bar{\mathbf{C}}$.

The case $c = 1$ is the half-plane version of the well-known criterion of Nehari [4] and Ahlfors and Weill [2]. As Ahlfors remarks [1, p. 29], the criterion (1.1), depending as it does on establishing that the values of $y^2 S(f,z)$ lie in a variable disk, seems too complicated to be useful. Ahlfors let $t \to \infty$ in (1.1) and asked if

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