ON THE RADIAL LIMITS OF FUNCTIONS WITH HADAMARD GAPS

D. Gnuschke and Ch. Pommerenke

To George Piranian, on the occasion of his retirement

1. Introduction and results. We consider functions f with *Hadamard gaps*, i.e.

$$f(z)=\sum_{k=0}^{\infty} a_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \geq \lambda > 1 \quad (k=0,1,\dots), \tag{1.1}$$

that are analytic in the unit disk $\mathbf{D}$. Let

$$M(r)=\max_{|z|=r}|f(z)| \quad (0 \leq r < 1) \tag{1.2}$$

and let $\dim E$ denote the Hausdorff dimension, i.e.

$$\dim E = \inf\{\delta : E \text{ has } \delta\text{-dimensional Hausdorff measure } 0\}.$$

It is clear that $0 \leq \dim E \leq 1$ for $E \subset \partial\mathbf{D}$.

If (a_k) is bounded then f is a normal function. Hence angular limits, radial limits and asymptotic values are the same by the Lehto–Virtanen theorem [14, p. 268]. On the other hand, if (a_k) is unbounded then f is not a normal function [15], and Murai [13] (see also [6]) has proved that f has the asymptotic value ∞ at every point of $\partial\mathbf{D}$.

We shall consider the radial behaviour at points ζ of $\partial\mathbf{D}$. If $\sum |a_k| = \infty$ then

$$\operatorname{Re} f(r\zeta) \to +\infty \quad \text{as } r \to 1-0 \tag{1.3}$$

holds on a set E with $\dim E > 0$ if $\lambda > 3$ and with $\dim E = 1$ if $n_{k+1}/n_k \to \infty$; see MacLane [11] and Hawkes [7, p. 28].

On the other hand, Csordas, Lohwater and Ramsey [5] have shown that, for any $\lambda > 1$,

$$\sum_k |a_k| = \infty, \quad (a_k) \text{ bounded} \tag{1.4}$$

implies that (1.3) holds on a set E of positive capacity which also has positive Hausdorff dimension. Their proof is based on results of Kahane, Weiss and Weiss [9], and the same is true of the following generalization.

THEOREM 1. *For $\lambda > 1$, there are positive numbers α, β, γ with the following property: If f has the form* (1.1) *and if*

$$\sum_k |a_k| = \infty, \quad \frac{|a_k|}{|a_0| + \cdots + |a_k|} \leq \alpha \quad (k \geq l), \tag{1.5}$$

then there is a closed set $E \subset \partial\mathbf{D}$ with $\dim E \geq \beta$ such that

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