ON REMOVAL OF PERIODS OF CONJUGATE FUNCTIONS IN MULTIPLY CONNECTED DOMAINS

D. Khavinson

1. Introduction. Let G be a domain in **C** bounded by n analytic Jordan curves $\gamma_1, \dots, \gamma_n$, $\Gamma = \bigcup_1^n \gamma_j$. Recall the following classical result of M. Heins—see [9].

For each $(n-1)$-tuple of real numbers $\Lambda_1, \dots, \Lambda_{n-1}$ there exist points $\zeta_1, \dots, \zeta_n$ on Γ and a positive harmonic function $u(z)$ in G such that $u(\zeta) = 0$ for all $\zeta \in \Gamma \setminus \{\zeta_1, \dots, \zeta_n\}$, $u(\zeta_i) = +\infty$, $i = 1, \dots, n$ and the periods of the conjugate function of $u(z)$ along $\gamma_1, \dots, \gamma_{n-1}$ equal to $\Lambda_1, \dots, \Lambda_{n-1}$ respectively.

The various refinements and applications of this result can be found in [5], [8], [9]. Also, see [5], [9], [10], [11] for the discussion concerning the corresponding statement for finite Riemann surfaces and its applications.

In this paper (in §3, Lemmas 1 and 2) we generalize the Heins result in the following sense.

For each $(n-1)$-tuple of real numbers $\Lambda_1, \dots, \Lambda_{n-1}$ and each positive Borel measure μ on Γ satisfying $\mu(\gamma_i) > 0$, $i = 1, \dots, n-1$ $(i = 1, \dots, n)$ there exist real (real positive numbers) $\lambda_1, \dots, \lambda_{n-1}$ $(\lambda_1, \dots, \lambda_n)$ such that the periods of the conjugate of the harmonic function defined by the Poisson integral of the measure $\tilde{\mu} : \tilde{\mu}|_{\gamma_i} \equiv \lambda_i \mu|_{\gamma_i}$, along $\gamma_1, \dots, \gamma_{n-1}$ are equal to $\Lambda_1, \dots, \Lambda_{n-1}$.

Let us give a brief description of the contents of the paper. In §2 we recall some basic facts of the function theory in multiply connected domains. For more details we send the reader to [3], [4], [13], [14].

In §3 we prove Lemmas 1 and 2. In §4, using Lemmas 1 and 2, we construct the analogs of the Schwarz kernel for the multiply connected domain G which allow us to reproduce analytic functions in G by means of the boundary values of their real parts. These kernels are different from those constructed in [2], [13], [16].

Finally, in §5 we consider certain applications of the results obtained in the previous sections. In particular, we show the existence in multiply connected domains of an analytic function in a given class (e.g., Nevanlinna's class, Hardy classes, etc.) with prescribed modulus of boundary values.

This problem has been studied in [7], [15]. Also, see [8, Ch. 4, §4]. But the functions constructed there are essentially different from those we obtain here.

Unfortunately, we have been unable to obtain an appropriate generalization of the results described above to finite Riemann surfaces. The fact is that the statements analogous to Lemmas 1 and 2 on a Riemann surface are much more complicated. The reason for that is that there are two kinds of periods on a surface, that is, periods along the boundary curves and around the handles. Therefore, one cannot expect formulas for the Schwarz type kernels on the surfaces to be as simple and clear as in multiply connected domains.

Received April 28, 1984. Final revision received June 16, 1984.
Michigan Math. J. 31 (1984).