TENSOR PRODUCTS OF REFLEXIVE SUBSPACE LATTICES

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Is the tensor product of two reflexive subspace lattices again reflexive? The dual question, whether the tensor product of reflexive operator algebras is reflexive, has been the subject of much recent investigation. This note will address the lattice question and the relationship between the two problems. Several special cases in which the lattice question has an affirmative answer will be discussed. In particular, the answer is affirmative if the subspace lattices are the full projection lattices of two injective von Neumann algebras.

Throughout this paper, all Hilbert spaces are separable and all projections are orthogonal projections. For a set $\mathcal{A}$ of bounded operators on $\mathcal{H}$ and a set $\mathcal{L}$ of orthogonal projections, we use the standard notation, Lat $\mathcal{A}$ and Alg $\mathcal{L}$, to denote the lattice of all projections left invariant by each operator in $\mathcal{A}$ and the algebra of all operators which leave invariant each projection in $\mathcal{L}$. Lattices and algebras which satisfy $\mathcal{L} = \text{Lat Alg } \mathcal{L}$ and $\mathcal{A} = \text{Alg Lat } \mathcal{A}$ are called *reflexive.* Reflexive algebras form a subclass of the class of weakly closed algebras and reflexive lattices form a subclass of the class of subspace lattices. (A *subspace lattice* is a lattice of projections which contains 0 and I and which is closed in the strong operator topology.) If a subspace lattice $\mathcal{L}$ consists of mutually commuting projections, it is called a *commutative subspace lattice* (CSL) and the corresponding algebra, Alg $\mathcal{L}$, is called a *CSL-algebra.* By a result in [1], every commutative subspace lattice is reflexive.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two weakly closed algebras, $\mathcal{A}_1 \otimes \mathcal{A}_2$ will denote the weakly closed algebra generated by all elementary tensors $A_1 \otimes A_2$, where $A_i \in \mathcal{A}_i$. When needed, the algebra generated by the elementary tensors (the algebraic tensor product) will be denoted by $\mathcal{A}_1 \odot \mathcal{A}_2$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices, $\mathcal{L}_1 \otimes_s \mathcal{L}_2$ will denote the smallest subspace lattice which contains all elementary tensors $P_1 \otimes P_2$, where $P_i \in \mathcal{L}_i$.

The reflexivity of the tensor product of two reflexive algebras would be assured if a stronger result, the algebra tensor product formula,

$$\text{(ATPF)} \qquad \text{Alg } \mathcal{L}_1 \otimes \text{Alg } \mathcal{L}_2 = \text{Alg}(\mathcal{L}_1 \otimes_s \mathcal{L}_2),$$

were known to be true. Similarly, the analogous problem for reflexive lattices would follow from a lattice tensor product formula,

$$\text{(LTPF)} \qquad \text{Lat } \mathcal{A}_1 \otimes_s \text{Lat } \mathcal{A}_2 = \text{Lat}(\mathcal{A}_1 \otimes \mathcal{A}_2).$$

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