EIGENVALUE ESTIMATES FOR CERTAIN NONCOMPACT MANIFOLDS

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1. Introduction. Suppose that X is a complete Riemannian manifold. The Laplacian Δ of X is essentially self adjoint on the space of smooth compactly supported functions. This means that Δ has a unique self adjoint extension to L^2X. In general, Δ may have both point and continuous spectrum. We say that ω is an eigenvalue of Δ if there exists a square integrable $\phi \in L^2X$ with $\Delta\phi = \omega\phi$. The symbol $N(\lambda)$ will denote the number of eigenvalues of Δ, which are less than λ.

For a general noncompact manifold X, standard techniques do not yield any estimate of $N(\lambda)$. In particular, the usual Neumann comparison only applies if λ is below the essential spectrum of Δ. The presence of continuous spectrum also causes serious difficulties in applying the heat kernel method.

In this paper, we study two specific classes of noncompact Riemannian manifolds. These are the manifolds with cylindrical ends, and the manifolds whose ends are isometric to the ends in locally symmetric spaces of rational rank one. Using the explicit metric structure on the ends of these manifolds, we employ a modified Neumann comparison to estimate $N(\lambda)$. The main result is the following.

THEOREM 1.1. *Suppose that X is a complete Riemannian manifold having a finite number of ends. Moreover, assume that either* (i) *each end is cylindrical or* (ii) *each end is isometric to an end in a locally symmetric space of Q-rank* 1. *Then $N(\lambda)$ has at most polynomial growth in λ.*

If $X = K\backslash G/\Gamma$ is a locally symmetric space of Q-rank one, then it is also interesting to consider the Casimir operator acting on a non-trivial K-type. Our method extends easily to prove the following.

COROLLARY 1.2. *Let X be a locally symmetric space of Q-rank one. Suppose that $N(\lambda)$ is the number of eigenvalues of the Casimir operator, belonging to a fixed K-type, which are less than λ. Then $N(\lambda)$ has at most polynomial growth in λ.*

Part (ii) of Theorem 1.1 resolves the trace class dilemma, for Q-rank one, as formulated by Borel and Garland [4] and Osborne and Warner [11, 12]. These authors proved that $N(\lambda)$ is finite for fixed λ. That is, there are no accumulation points of the set of eigenvalues. However, their method, which is based on the theory of Eisenstein systems, only yielded a growth estimate when X is a locally symmetric space of real rank one. The importance and applications of our bound on $N(\lambda)$ are clearly described in [11] and [12].

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