NONCOMPACT RIEMANNIAN MANIFOLDS WITH PURELY CONTINUOUS SPECTRUM

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1. Introduction. Let (M^n, ds^2) be an n-dimensional Riemannian manifold with Laplacian Δ, where

$$\Delta u = \frac{1}{\sqrt{g}} \Sigma_{ij} \left(\frac{\partial}{\partial x^i} \sqrt{g} g^{ij} \frac{\partial}{\partial x^j} u \right)$$

in local coordinates. Here $ds^2 = \Sigma_{i,j} g_{ij} dx^i dx^j$, $g = \det(g_{ij})$, and $(g^{ij}) = (g_{ij})^{-1}$. It is well-known that if M^n is compact then the spectrum of Δ is discrete ([5]). On the other hand, if M^n is noncompact the spectrum may be purely continuous, as it is for Euclidean space (and hyperbolic space, see [9]), purely discrete (see [4], [11], [16]), or a mixture of the two types, possibly with eigenvalues embedded in the continuous spectrum (cf. [10]).

It was first shown by Pinsky [20] (for rotation invariant metrics) and then by Donnelly [10] (for general metrics) that if M is simply-connected and has curvature $K(r,\theta) \le -k^2$ with $K \to -k^2 < 0$ and angular derivatives K_θ and $K_{\theta\theta} \to 0$ (all with sufficient speed) as $r \to \infty$, then the Laplacian has no eigenvalues (i.e., the spectrum is purely continuous as it is for the constant curvature case $K \equiv -k^2 < 0$). In this paper we prove an analogous result for manifolds with $K \to 0$, involving only conditions on K and K_θ. Our method is related to Rellich's original work on this problem for Euclidean space [23], while Donnelly uses a modification of Kato's extension [17] of Rellich's work to operators with variable coefficients. The general Rellich–Kato procedure, which involves estimating the growth of integrals of solutions of the eigenvalue equation (in this case $\Delta f + \lambda f = 0$) has already proved itself useful in mathematical physics (cf. [1], [25]).

It is a pleasure to thank M. Pinsky and H. Donnelly for sending us preprints of their work. As in [10] and [20], we write out the proofs only for two-dimensional surfaces for simplicity of exposition. There is a known technique for translating these results to manifolds of dimension ≥ 3 which has already been sketched in [10].

In discussing eigenvalues (as opposed to the continuous spectrum) it is important to fix a specific self-adjoint extension of a given symmetric operator. For this reason we give, in Section 2, a quick proof of Gaffney's result [12] concerning the essential self-adjointness of Δ.

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