CYCLIC VECTORS OF BOUNDED CHARACTERISTIC IN BERGMAN SPACES

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1. Introduction. Let D be the unit disk in the complex plane. H^p denotes the usual class of functions analytic on D. Let A^2 be the Bergman space of analytic functions f such that

$$\|f\|_{A^2}^2 = \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^2 r\, dr\, d\theta < \infty.$$

If $f \in A^2$, let $[f]$ denote the smallest closed subspace of A^2 which contains $\{z^n f\}_{n=0}^{\infty}$. If S is the unilateral shift $Sf = zf$, then $[f]$ is the smallest closed subspace of A^2 containing f which is invariant under S.

A function $f \in A^2$ is called *cyclic* if $[f] = A^2$. It is an open problem to give a useful characterization of the cyclic functions in A^2. On the other hand, if $f \in H^1$ much more is known.

Recall that any function $f \in H^1$ has the canonical factorization $f = BsF$, where B is a Blaschke product, F is an outer function and s is a singular inner function generated by a singular measure σ:

$$s(z) = s_\sigma(z) = \exp\left(-\int_T \frac{\zeta + z}{\zeta - z}\, d\sigma(\zeta)\right).$$

Here, σ is singular with respect to Lebesgue measure on the unit circle T.

A closed set $K \subseteq T$ of Lebesgue measure 0 is called a *BCH set* (Beurling–Carleson–Hayman) if

$$\int_T \log \frac{1}{\rho_K(\zeta)} |d\zeta| < \infty,$$

where $\rho_K(\zeta) = \inf_{z \in K} |z - \zeta|$. Equivalently, K is a BCH set if (i) $|K| = 0$, and (ii) $\sum |I_k| \log(1/|I_k|) < \infty$, where $|E|$ denotes the Lebesgue measure of E and $T \setminus K = \bigcup_{k=1}^{\infty} I_k$ is the canonical decomposition of $T \setminus K$ into disjoint open arcs.

We can now state the following theorem.

THEOREM I. *If f is in H^1 and non-vanishing, then f is cyclic for A^2 if and only the singular factor of f equals s_σ where $\sigma(K) = 0$ for all BCH sets K.*

The necessity is due to H. S. Shapiro [13]. The sufficiency was established independently by Korenblum [9, 10] and Roberts [14].

Actually the authors above proved necessity and sufficiency in case $f = s_\sigma$. The extension to H^1 was noted in [4]. We sketch a proof. If $f \in H^1$, then $\|f\|_{A^2} \leqslant$

Received September 8, 1983. Final revision received February 21, 1984.

The first author was supported in part by a grant from Wayne State University. The second and third authors were supported in part by the National Science Foundation.

Michigan Math. J. 31 (1984).