ON THE ZEROS OF JONQUIÈRE'S FUNCTION WITH A LARGE COMPLEX PARAMETER

Wolfgang Gawronski and Ulrich Stadtmüller

1. Introduction and summary. In this paper we are dealing with Jonquière's function (cf. [10, p. 33], [15, p. 364], [18, p. 280]) defined by its power series

$$f_\kappa(z) := \sum_1^\infty n^\kappa z^n, \quad \kappa = \kappa_0 + i \cdot \kappa_1 \in \mathbf{C} \tag{1}$$

for $|z| < 1$; by analytic continuation it is seen to be holomorphic in the cut plane

$$\mathbf{C}^* := \{z \in \mathbf{C} \mid \text{If } \operatorname{Re} z \geqq 1, \text{ then } \operatorname{Im} z \neq 0\}. \tag{2}$$

If $\kappa = k$ is a positive integer, then f_k is connected with the geometric series by the simple relation

$$f_k(z) = \left(z \frac{d}{dz}\right)^k \frac{1}{1-z}$$

[14, p. 7, problem 46]. Moreover, Jonquière's function is of some significance in various parts of mathematics and physics. For instance, it occurs in analytic number theory [8] as a generalization of Riemann's ζ-function, in summability theory concerning equivalence problems for Césaro and certain discontinuous Riesz means [13, ch. IV, 3], and in research on the structure of polymers [17]. Questions in Riesz summability, especially, require the number and the location of the zeros of f_κ in $\mathbf{C}^*$ when κ is *real*. The first complete result for this case is due to A. Peyerimhoff [12] stating that all zeros in $\mathbf{C}^*$ are real and $\leqq 0$. Moreover, they have order one and their exact number is $k+1$ if $k < \kappa \leqq k+1$, $k \in \mathbf{N}_0$, and 1 if $\kappa \leqq 0$. Different and modified proofs as well as the dependence of the zeros on the real parameter κ were given in a series of papers [2, 3, 4, 5, 6, 7, 11, 12, 16, 19]. In continuation of these investigations we ask the following questions.

(i) In case of *real* κ, how are the zeros distributed on the negative real axis if κ becomes large?

(ii) What can be said about the zeros of f_κ when κ is *complex*?

In view of the close relation of f_κ with Riemann's ζ-function (observe that $f_\kappa(-1) = (2^{\kappa+1} - 1)\zeta(-\kappa)$) the second question without any restriction for κ includes the problem of finding all complex zeros of ζ. Fornberg and Kölbig [1] investigated the zeros of $f_\kappa(x)$ in the half-plane $\{\kappa \in \mathbf{C} \mid \operatorname{Re} \kappa < 0\}$, for fixed $x \in (-1, 1)$. Their considerations are restricted to the behaviour of these zeros when $x \to 0$ and $x \to 1^-$. The latter case is used to get a numerical approach to the zeros of the ζ-function. We are interested in the zeros of $f_\kappa(z)$ in the z-plane for fixed $\kappa \in \mathbf{C}$. Treating the first question above, it turns out that most of the arguments used there are also valid for complex κ, and that we can obtain good

Received August 15, 1983. Final revision received May 5, 1984.
Michigan Math. J. 31 (1984).