REPRESENTATIONS OF THE DISCRETE HEISENBERG GROUP AND COCYCLES OF AN IRRATIONAL ROTATION

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Introduction. One of the most familiar examples in ergodic theory is that of an irrational rotation of a circle. These rotations are well understood in many ways, but continue to be studied in their appearances outside ergodic theory proper. For example, we mention their uses in C^*-algebras [4, 25, 26, 30], function algebras [10, 12] and unitary representations of groups [15, 16, 22, 23, 28]. The main purpose of this paper is to present some information about (one-) cocycles (defined below) of irrational rotations in the context of representation theory, and to raise a few questions in that same context.

The existence of irreducible cocycles of every dimension is known from general arguments [28, 32, 35], and explicit formulas have been given in some cases [1, 5, 13, 15]. Our interest is in specific formulas, just as specific formulas are sought when studying group representations. Indeed, we derive cocycle formulas from formulas for representations of the discrete Heisenberg group, and then use the cocycles to get formulas for representations of the Mautner group. Thus we are closely tied to representation theory in our motivation and in our method. Besides the explicitness of the cocycle formulas, we want to emphasize that they arise in a systematic way in the representation theory setting.

Cocycles appear in the representation theory of locally compact groups (taken here to be second countable) when forming induced representations. Let G be a locally compact group with a (right) Borel action on a compact metric space X. If $\mathcal{U}$ is the unitary group of a Hilbert space $\mathcal{H}$, a Borel function $S: X\times G \to \mathcal{U}$ such that $S(x, g_1 g_2) = S(x, g_1) S(xg_1, g_2)$ for $x\in X$, $g_1, g_2 \in G$ is called a cocycle for the action. If $X\times G$ is given its natural groupoid structure, such an S is a homomorphism of $X\times G$ to $\mathcal{U}$, i.e. a representation of $X\times G$ [22, 23, 27, 28]. Suppose μ is a finite Borel measure on X, quasi-invariant and ergodic for the action of G, and ν is a finite measure equivalent to Haar measure on G. By an *intertwining* between two cocycles S and S' for the action of G on X, relative to μ, we mean an operator-valued function A on X such that $A(x)S(x,g) = S'(x,g)A(xg)$ for $\mu\times\nu$ almost all pairs (x,g) in $X\times G$. Two such cocycles S and S' are *equivalent* if there exists a unitary-valued intertwining between them, and a single cocycle S is called *irreducible* (relative to μ) if the only intertwinings between S and itself are the scalar operator-valued functions.

The groups of interest to us here are semidirect products $G = NK$ where N is normal and abelian. Via inner automorphism, there are actions of both G and K

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