THE σ-REGULAR REPRESENTATION OF $\mathbf{Z}\times\mathbf{Z}$

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Let $\mathbf{Z}$ denote the group of integers. There exist multipliers σ on $\mathbf{Z}\times\mathbf{Z}$ such that the group extension of $\mathbf{Z}\times\mathbf{Z}$ by σ is a non-Type I group. In fact, the σ-regular representation of such a lattice group is a Type II_1 factor; the consequences of this fact were investigated by Pukanszky in [5]. The main result of this paper is the existence of decompositions of the σ-regular representation of $\mathbf{Z}\times\mathbf{Z}$ with respect to an infinite family of mutually disjoint measures. The integrands in the decompositions are induced irreducibles; furthermore, they can be canonically chosen so that the restrictions to two given normal subgroups are associated with Lebesgue measure quasi-orbits on tori with arbitrary finite relatively prime multiplicities.

Let G be a locally compact group, T the circle group. A *multiplier* (or cocycle) on G is a Borel function $\sigma: G\times G\to T$ satisfying $\sigma(a,b)\sigma(ab,c)=\sigma(a,bc)\sigma(b,c)$ and $\sigma(a,e)=\sigma(e,a)=1$ for all $a,b,c\in G$, where e is the identity of G. Two multipliers σ and σ' are *similar* if there is a Borel $\beta: G\to T$ such that $\sigma'(a,b)=\beta(a)\beta(b)\beta(ab)^{-1}\sigma(a,b)$ for all a,b. A multiplier similar to unity is called a coboundary. For $G=\mathbf{Z}\times\mathbf{Z}$, we find that every multiplier is similar to one of the form $\exp(iB)$, where B is a real bilinear form on $G\times G$. This follows from [4, Theorem 9.6] and the fact that every multiplier on a cyclic group is a coboundary. For convenience, we will adopt the following conventions. Elements of G will be denoted either by n or by (p,q), with subscripts as needed. We regard T as $R/2\pi\mathbf{Z}$, elements typically denoted by u, w. We view elements of T^2 as vectors V or (V_1, V_2), with group action written additively. Finally, let e_1, e_2 be the usual basis vectors in the real plane, $e_3=e_1+e_2$; $\langle\,,\rangle$ will denote the usual inner product.

For a given multiplier σ on $G=\mathbf{Z}\times\mathbf{Z}$, define the *$\sigma$-regular representation R^σ* by the formula $(R_g^\sigma f)(g')=\sigma(g',g)f(g'g)$ for $f\in L^2(G)$. If $F: L^2(G)\to L^2(T^2)$ is the Fourier transform, define $\hat{R}^\sigma=FR^\sigma F^{-1}$. We wish next to define an action on $\hat{R}^\sigma$ by certain homomorphisms of T^2. To this end, let $M=\begin{pmatrix} a_1 & a_2\\ a_3 & a_4\end{pmatrix}\in SL(2,\mathbf{Z})$. It is well known that M induces a measure-preserving homomorphism of T^2, and hence a unitary operator V_M on $L^2(T^2)$, given by $(V_M\phi)(v)=\phi(Mv)$. We will say M acts on $\hat{R}^\sigma$ by $M\cdot\hat{R}^\sigma=V_M\hat{R}^\sigma V_M^{-1}$. To compute the effect of this action, fix a real matrix $A=\begin{pmatrix} t_1 & t_2\\ t_3 & t_4\end{pmatrix}$, and let σ be given by $\sigma(n_1,n_2)=\exp(i\langle n_1, An_2\rangle)$. Then, for all $\phi\in L^2(T^2)$,

$$\hat{R}_n^\sigma\phi(v)=c(n)\exp(-i\langle v,n\rangle)\phi(v+An)$$

and

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