A CLASS OF WASSERSTEIN METRICS FOR PROBABILITY DISTRIBUTIONS

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0. Introduction. There are several natural metrics that one can place on spaces of probability distributions (or "laws"). These include total variation, Prohorov's ρ metric, dual norms induced by spaces of Lipschitz functions, and the so-called Wasserstein distance. A discussion of these is to be found in Dudley [4], especially in Lectures 8, 18, and 20. See also [3].

The Wasserstein metric seems to have arisen first in connexion with the transport of mass problem. In a certain form, this dates back to 18th-Century work of Monge, but perhaps the first significant modern research was due to Kantorovich [8]. The realisation that the Wasserstein metric can be taken as a reasonable distance on spaces of random variables or probability distributions was first expressed in a paper of Kantorovich and Rubinstein [9], where the problem is put in the context of infinite-dimensional linear programming, and a duality theorem is proposed. This line of thought continues in Kemperman [10]. A general, abstract context for the metric is to be found in Szulga [16].

Although natural and far-reaching as a theoretical tool, the Wasserstein metric has a definite drawback: explicit calculation is difficult for most concrete examples. For distributions on the line, the problem is not severe, and there is a result of Vallander [17] to cover this case. In some unpublished work of Neveu and Dudley, the suggestion was made that a somewhat altered (L^p) version of the Wasserstein be considered. The present paper contains a calculation of the L^2 Wasserstein distance between arbitrary n-dimensional Gaussian distributions. The problem can be reduced to a Lagrange multiplier optimisation: this calculation forms §2 of the paper. Section 1 presents some general results concerning the family of L^p Wassersteins for $1 \leqslant p \leqslant \infty$, whereas §3 concludes with a few open questions and speculations.

1. The L^p Wasserstein metrics. Throughout this section, (S, d) represents a complete, separable metric (Polish) space and 0 a fixed but arbitrarily chosen point in S. For each p with $1 \leqslant p < \infty$, define $\mathfrak{M}_p = \mathfrak{M}_p(S)$ to be the collection of all probability measures (i.e. laws) P on (the Borel sets of) S for which

$$\int_S d^p(X, 0)\, dP(X)$$

is finite. Let $\mathfrak{M}_\infty(S)$ be the set of all laws on S with bounded support. It is easy to show that the spaces $\mathfrak{M}_p$ do not depend on the choice of the point 0.

Let P_1 and P_2 be members of $\mathfrak{M}_p$ $(1 \leqslant p < \infty)$. The L^p Wasserstein distance between P_1 and P_2 is defined by

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