ASYMPTOTIC ESTIMATES FOR THE PERIODS OF PERIODIC SOLUTIONS OF A DIFFERENTIAL-DELAY EQUATION

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Introduction. A periodic solution $x(t)$ of

$$x'(t) = -\alpha f(x(t-1)) \tag{0.1}$$

will be called a "slowly oscillating periodic solution (of (0.1))" if there exist numbers $q>1$ and $\bar{q}>q+1$ such that $x(t)>0$ for $0<t<q$, $x(t)<0$ for $q<t<\bar{q}$, and $x(t+\bar{q})=x(t)$ for all t. The word "slowly" refers to the fact that the separation of zeros of $x(t)$ is greater than the time lag, which is 1.

If $f(x)$ is odd (the case considered in this paper), it is useful to consider a subclass of the slowly oscillating periodic solutions of (0.1). A slowly oscillating periodic solution of (0.1) is called an *S-solution* (in the notation of D. Saupe [7, 8]) if $\bar{q}=2q$ and $x(t+q)=-x(t)$ for all t. Actually, it will be useful to be more pedantic and call an S-solution of (0.1) a pair (α, x) such that $x(t)$ is a periodic solution of $x'(t)=-\alpha f(x(t-1))$, $x(t)$ is positive on an interval $(0,q)$ where $q>1$, and $x(t+q)=-x(t)$ for all t. This paper will treat properties of S-solutions of (0.1) and in particular properties of the maps $(\alpha, x) \to q = q(\alpha, x)$ for α large and (α, x) an S-solution.

The problem of the existence and qualitative properties of slowly oscillating periodic solutions of (0.1) has been studied by several authors, and there is ample evidence by now that the qualitative properties of periodic solutions of (0.1) may depend subtly on the function f. Here we shall consider odd functions $f(x)$ which are similar to $f_r(x)$, where

$$f_r(x) \equiv x(1+|x|^{r+1})^{-1}.$$

Equation (0.1) with such an f was suggested by J. Yorke (in a private communication to the second author) as a model for somewhat more complicated-looking equations like

$$x'(t) = -Ax(t) + Bf(x(t-1)), \quad A, B>0, \tag{0.2}$$

which had been proposed by Mackey and Glass [2, 3] in connection with physiological control theory.

D. Saupe [7, 8] has carried out a careful numerical study of equation (0.1) for $f(x)=f_r(x)$. Saupe's results suggest that (0.1) displays very complex dynamical behaviour, but little has been proved. It has, however, been proved that if $r>2$ and α is sufficiently large, then equation (0.1) has at least three S-solutions (Saupe's numerical studies actually suggest the existence of at least seven

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