DECOMPOSITIONS AND APPROXIMATE FIBRATIONS

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1. Introduction. In this paper we investigate upper semi-continuous (= u.s.c.) decompositions $\mathcal{G}$ of manifolds M (without boundary) into continua having the shape of closed manifolds of some fixed dimension $k>0$. The fundamental problem considered is the extent to which the decomposition map $p: M \to M/\mathcal{G}$ is an approximate fibration. Coram and Duvall [3] initiated this type of investigation when they considered decompositions of the 3-sphere into 1-spheres satisfying several additional restrictions, including that the decomposition space be the 2-sphere, and they showed that the decomposition map was an approximate fibration over the complement of at most two points. Our main result in §2 (Theorem 2.10) is that if $\mathcal{G}$ is an u.s.c. decomposition of M as above and if the decomposition space is finite dimensional, then there exists a dense open subset $U \subseteq M/\mathcal{G}$ such that the restriction $p \mid p^{-1}(U): p^{-1}(U) \to U$ is an approximate fibration. From §3, it follows that U is a generalized manifold.

Daverman [5] showed that if the dimension of M is $k+1$ then $M/\mathcal{G}$ is a 1-dimensional manifold, and if each element of $\mathcal{G}$ is a locally flat submanifold of M then p is an approximate fibration, provided $M/\mathcal{G}$ has empty boundary. He also constructed examples to show that either local flatness or some condition on the relationship between the fundamental groups of M and the elements of $\mathcal{G}$ is needed in order to show that p is an approximate fibration. Supposing that each element of $\mathcal{G}$ has the shape of a closed k-dimensional manifold, and that $M/\mathcal{G}$ is, homeomorphic to $\mathbf{R}^1$, then we show (Theorem 5.15) that the inclusion of each element of $\mathcal{G}$ into M is a homology equivalence; this generalizes Lemma 6.2 of [5]. Furthermore, if the inclusion-induced $\check{\pi}_1(g) \to \pi_1(M)$ is an isomorphism for each $g \in \mathcal{G}$ and the integral group ring of $\pi_1(M)$ is Noetherian, then we show (Theorem 5.16) that p is an approximate fibration.

The value of knowing that p is an approximate fibration, for example in the latter case, is that p can be approximated by locally trivial bundle maps and hence M can be expressed as a product $N \times \mathbf{R}$, where N is a closed k-manifold which has the shape of the elements of $\mathcal{G}$.

We refer the reader to [2] for the definition of approximate fibrations and their properties. We use the Mardešić–Segal approach to shape theory [11], although we use Borsuk's terminology of FANR (= fundamental absolute neighborhood retract) rather than Mardešić–Segal's term ASNR (= absolute shape neighborhood retract). A fundamental property of FANR's which we often employ in this paper is that if $\{U_i\}_{i=1}^{\infty}$ is a nested sequence of neighborhoods of an FANR X in an ANR such that $\bigcap_{i=1}^{\infty} U_i = X$, then the induced inverse systems of homology and homotopy groups, $\{H_*(U_i)\}_{i=1}^{\infty}$ and $\{\pi_*(U_i)\}_{i=1}^{\infty}$, are stable; that

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