ACTIONS OF SU(2) ON S^7

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1. Introduction. It is well known that SU(2) can act freely on S^7. This paper is concerned with *almost free* actions. An action of a compact Lie group is called *almost free* if every isotropy group is finite, which implies that every orbit has the same dimension as the group. Although SO(3) cannot act freely on S^7, Oliver [5] has shown that SO(3) can act almost freely on S^7. (This shows that a theorem in [2] is false.) The group SO(3) is the quotient by the center C of SU(2), and C consists of 2 elements, namely ± 1 if SU(2) is considered as the group of unit quaternions. Thus Oliver's work shows that there is a non-free but almost free action of SU(2) on S^7. However this action is not effective. The main result of this note is as follows.

THEOREM 1. *Let $G = \mathrm{SU}(2) = S^3$ act smoothly, effectively, and almost freely on S^7. If the fixed point set of the center C is an empty set, that is $F(C) = \varnothing$, then the action is free.*

This theorem may possibly be true in dimension $4k-1$, $k \geqslant 2$.

An example of an *effective* almost free action of SU(2) on S^7 may be constructed at least topologically. In order to do this take the join of SU(2) acting on itself and SU(2) acting on SU(2)/I where I is the doubled icosahedral group. As a space the join is S^7 by the double suspension theorem (see e.g., Cannon, Bull. Am. Math. Soc., 84, 1978, pp. 832–866), and $F(C) = \mathrm{SU}(2)/I$. This may be the only example with $F(C) \neq \varnothing$.

As will be seen the proof amounts essentially to showing that a certain kind of action of N (N = normalizer of the circle group) on a mod p 3-sphere cannot be extended to an action of SU(2) in the way which would be required.

2. General remarks. This section lists two facts about general actions of compact Lie groups which will be useful in this paper. It is assumed that G is a compact Lie group acting on a manifold M with base space M^* and projection $\pi: M \to M^*$.

I. *Let A be a closed connected subset of M which is a cross-section of the orbits it touches. Let all isotropy groups G_x, $x \in A$, be conjugate. Then $G(A)$ is a topological product $G(A) = A \times G/G_a$ for any fixed $a \in A$.*

Proof. For any $x \in A$ there is a homeomorphism $G(a) \to G(x)$ given by $gG_a(a) \to gG_x(x)$, $g \in G$. This determines the desired homeomorphism because G_x, $x \in A$, varies in a continuous way [4]. □

II. *If S^* is a closed path in M^* and if all orbits in S^* are of the same isotropy type, then there is a cross-section S of $\pi^{-1}(S^*)$ and $\pi^{-1}S^* = X \times G/G_a$, for any*

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