GENERATING NON-NOETHERIAN MODULES EFFICIENTLY

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In their excellent 1973 paper [2], Eisenbud and Evans developed a unified treatment of a number of results relating the dimension of a ring to the generating sets of modules over that ring. Throughout, they assume the ring in question has Noetherian j-spectrum. Subsequent work by the present author [3], Vasconcelos and Wiegand [5], and Brumatti [1] has produced some of these results in a non-Noetherian setting, but many questions remained unanswered. In fact, with a few fairly minor modifications, virtually all of the results in [2] can be demonstrated without any Noetherian assumption. Moreover, the same unified presentation can be followed.

In §1, we shall introduce the notation we shall use and prove some elementary lemmas which shall be needed later. In §2, we present the results. This begins with a generalization of Bass's Stable Range Theorem (Theorem 2.1), which immediately allows us to extend Kronecker's Theorem that radical ideals (that are radicals of finitely generated ideals) are radicals of $(\dim R+1)$-generated ideals (Corollary 2.4). More importantly, (2.1) serves as the fundamental lemma needed to prove our version (Theorem 2.5) of the Basic Element Theorem [2, Theorem A, p. 282]. With this, we may extend the "corollaries" of Theorem A—Serre's Theorem (2.6), Bass's cancellation theorem (2.7), and the Forster–Swan Theorem (2.8, 2.9). In §3, we offer a few examples to illustrate the necessity of some of the modifications which have been made in the presentation.

The methods employed herein are not really new; primarily they are descended from the techniques introduced in [3]. The presentation is quite different however and no familiarity with the earlier paper will be required.

1. Throughout, R will be a commutative ring with identity and A will be a finite R-algebra (meaning finitely generated as an R-module). On first reading, the simplifying assumption $A=R$ may be helpful. All modules are unitary left A-modules.

Let $\mu(A,M)$ denote the minimal number of generators of M as an A-module. Following [2], we say a submodule $M'\subset M$ is basic at a prime P of R if $\mu(A_P,(M/M')_P)<\mu(A_P,M_P)$, and is t-fold basic if $\mu(A_P,(M/M')_P)\leqslant\mu(A_P,M_P)-t$. We say a set $m_1,\dots,m_u\in M$ is basic (resp. t-fold basic) at P if $A(m_1,\dots,m_u)$ is. We also use the terminology basic (resp. j-basic, X-basic) to mean basic at every prime $P\in\operatorname{Spec}R$ (resp. j-spec R, X).

We make frequent use of Spec R, the set of prime ideals of R with the usual Zariski topology. We will also need the patch topology; this has the same points as Spec R but has for a closed subbasis the Zariski-closed and Zariski quasi-

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