A MATHEMATICAL MODEL FOR A WAKE

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1. Introduction. This paper concerns a type of free boundary problem which gives a mathematical model for a wake around an obstacle in the flow of a two-dimensional fluid which is incompressible, irrotational and inviscid. To formulate the problem, we begin with an infinite striplike region $S=\{x+iy: \phi_1(x)<y<\phi_2(x), -\infty<x<\infty\}$, ϕ_1 and ϕ_2 continuous. If α is a Jordan curve with interior Δ contained inside of S, then a wake around α is a doubly connected region $\omega\subset S$ such that $\Delta\cap\omega=\varnothing$, $\partial\omega=\partial S\cup\gamma$, where $\partial S\cap\gamma=\varnothing$, and there is a harmonic function V_ω in ω satisfying:

(a) $V_\omega(z)=0$ for $z\in\partial S$,
(b) $V_\omega(z)=1$ for $z\in\gamma$, and
(c) $|\text{grad } V_\omega(z)|=p$ for $z\in\gamma-\alpha$.

The number p is a constant. (See Figure 1.)

The set $\gamma-\alpha$ is made up of *free stream lines* [6] and is called the free boundary. We will show that there is a region ω when ∂S and α are starlike. The methods in [5] can be used to prove existence. In particular, a solution to this problem is obtained by considering

$$J(v)=\iint_{S-\Delta}(|\nabla v|^2+I_{(v>0)}p^2)\,dx\,dy,$$

where I_A is the characteristic function of A. If $K=\{v: v=1 \text{ on } \partial S,\ v\geqslant 0 \text{ in } S-\Delta\}$, then u_p, a minimum for $J(v)$, will give a wake around α by letting $\omega=\{u_p>0\}$ and $V_\omega=1-u_p$. However, Beurling's paper [4] and various qualitative results in [8], [9] and [10] can be used to get existence with the additional information that the free boundaries are starlike. Furthermore, we obtain values for the constant p where the solution will be non-degenerate, i.e., $\gamma-\alpha\neq\varnothing$. The main idea is to first formulate the problem for compact regions and then approximate S by a sequence of these compact regions. In the compact case we are able to use results in [1], [4], [8], [9] and [10] to find wakes. However, the results in these papers are formulated for doubly connected regions and must be extended to the simply connected case. Before beginning, we mention that wakes are also studied in [10], which includes a survey of classical methods to attack the problem of an infinite wake.

2. COMPACTNESS. Suppose D is simply connected on the compact Riemann sphere and Γ is the boundary of D which is compact in the open plane but does not reduce to a point. Let C be the class of all doubly connected regions $\omega\subset D$ such that $\partial\omega=\Gamma\cup\gamma$ where $\gamma\cap\Gamma=\varnothing$ and γ is compact in the open plane. We call γ the free boundary of ω. (See Figure 2.)

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