STABILITY PROPERTIES OF THE YANG-MILLS FUNCTIONAL NEAR THE CANONICAL CONNECTION

H. Turner Laquer

Introduction. Connections and curvature were introduced in the early part of this century by Cartan and Weyl, and by the 1950's these ideas were a well-established part of differential geometry ([9], [14]). In the past few years, this area of mathematics has again received widespread attention by both mathematicians and physicists in the form of Yang-Mills theory.

The Yang-Mills functional gives a measure of the total curvature of a connection in a principal bundle. The critical points of this functional, the so-called Yang-Mills connections, appear to play a fundamental role in physics. In §1, we define the functional and derive the corresponding variational equations for its critical points.

Because of the physical applications, much of the work in Yang-Mills theory has dealt with the case of principal bundles over four-dimensional manifolds ([2], [4], [7], [17]). The functional for bundles over Riemann surfaces has also been studied in some detail ([3]). In this paper, we are primarily concerned with the Yang-Mills functional for bundles of the form $P: G \to G/H$ and for associated principal bundles, $P_\lambda = G \times_H U \to G/H$, where $\lambda: H \to U$ is a Lie group homomorphism. These bundles have a canonical G-invariant connection, ω_0, and we are especially interested in the behavior of the functional near ω_0.

The key to doing explicit calculations on these homogeneous spaces is that geometric objects on G/H are given by sections of bundles which are associated to P by representations of H. The space of such sections becomes a G-module, called the induced representation, whose structure is given by Frobenius reciprocity (Theorem 2.1). In §2, we also develop the notation of equivariant functions which is used for subsequent calculations.

In §3, we study the Yang-Mills functional near the canonical connection. In particular, we show that ω_0 is Yang-Mills (Theorem 3.1) and we derive methods for computing the index and nullity at ω_0 (Theorems 3.3 and 3.4). These formulas involve Laplacians and Casimir operators and, in §4, these operators are related to representation theory. The index of a representation, originally introduced by Dynkin ([8]), plays an important role.

Finally, in §5 we consider some examples using Theorems 3.3 and 3.4. In particular, we determine the index and nullity of the Yang-Mills functional at the canonical connection for the bundles $G \to G/H$ when G/H is a compact irreducible Riemannian symmetric space. These results are given by Theorem 5.1 and Table II. The canonical connection gives a stable critical point in all cases except for spheres S^n when $n \geq 5$, compact simple Lie groups, quaternionic projective spaces $\mathrm{SP}(p+1)/\mathrm{SP}(p) \times \mathrm{SP}(1)$, and the exceptional symmetric spaces E_6/F_4

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