ON A HARDY AND LITTLEWOOD IMBEDDING THEOREM

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Introduction. For f in the class $H^p(\text{disc})$, $0<p<\infty$, Hardy and Littlewood [8, Theorem 31] showed that

$$\left\{\int_0^1 \left[(1-\rho)^{1/p-1/r}\left(\int_0^{2\pi} |f(\rho e^{i\theta})|^r\, d\theta\right)^{1/r}\right]^q (1-\rho)^{-1}\, d\rho\right\}^{1/q} \leqslant c\|f\|_{H^p},$$

where $0<p<r\leqslant\infty$, $p\leqslant q\leqslant\infty$. They used this inequality in their discussion of fractional integrals and convolutions of power series. The case $0<p<1=r=q$ has also been used by Duren, Romberg and Shields [2] to identify the bounded linear functionals on $H^p(\text{disc})$. Recently Flett [5] observed that the inequality gives easy proofs of a number of interesting results, and simplified its proof.

The purpose of this note is to present a simple proof of a general version of this inequality and to discuss some of its applications in various settings. We begin by introducing a maximal function. Let (X,μ) and (T,ν) be measure spaces with positive measures $d\mu$ and $d\nu$ respectively. Assume that to each $(x,t)\in X\times T$ we associate a μ-measurable set $B(x,t)\subseteq X$ so that the family $\mathcal{B}=\{B(x,t)\}$ verifies three conditions, namely

(i) $x\in B(x,t)$ for each $t\in T$;
(ii) if $y\in B(x,t)$, then $x\in B(y,t)$; and
(iii) $0\lneqq\mu(B(x,t))\lneqq\infty$.

For functions f defined on $X\times T$ and $x\in X$ we set

$$M_{\mathcal{B}}f(x)=\sup_{t\in T}\ \sup_{y\in B(x,t)} |f(y,t)|.$$

We begin by observing the following:

PROPOSITION. *Suppose $M_{\mathcal{B}}f\in L^p(X)$, $0<p<\infty$. Then*

$$|f(x,t)|\leqslant \min\left(M_{\mathcal{B}}f(x), \left(\frac{1}{\mu(B(x,t))}\int_{B(x,t)} M_{\mathcal{B}}f(y)^p\, d\mu\right)^{1/p}\right).$$

Proof. It is immediate. From (i) it follows that $|f(x,t)|\leqslant M_{\mathcal{B}}f(x)$, and from (ii) that $|f(x,t)|\leqslant \inf_{y\in B(x,t)} M_{\mathcal{B}}f(y)$, which in view of (iii) gives the desired conclusion at once. □

We can now prove our first imbedding result.

THEOREM 1. *Suppose that $M_{\mathcal{B}}f\in L^p(X)$, $0<p<\infty$, and that q and α verify $p\leqslant q<\infty$ and $-1+q/p\leqslant\alpha<q/p$. Furthermore, assume that the non-negative function $k(x,t)$ verifies*

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