POLYNOMIAL RINGS OVER A HILBERT RING

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The object of the present paper is to clarify the literature surrounding an incorrect result, Theorem 3.3 of [2], on just when polynomial rings in infinitely many variables over a Hilbert ring are again Hilbert.

A ring R (commutative with unity) is a Hilbert ring if every prime ideal of R is an intersection of maximal ideals. The concept of a Hilbert ring (also called a Jacobson ring) was introduced by Goldman in [4] and Krull in [9], where it is shown that if R is a Hilbert ring, then the polynomial ring $R[X]$ is again a Hilbert ring. In particular, the fact that for k a field, the polynomial ring $k[X_1,\dots,X_n]$ is a Hilbert ring yields a ring-theoretic formulation of the Hilbert Nullstellensatz. Krull also showed in [9] that a polynomial ring $k[\{X_i\}_{i=1}^{\infty}]$ in a countably infinite number of variables over a field k is a Hilbert ring if and only if the field k has uncountable cardinality. For $\{X_\lambda\}_{\lambda\in\Lambda}$ an infinite set of indeterminates, Gilmer in [2] considers the general question of when the polynomial ring $R[\{X_\lambda\}]$ is a Hilbert ring. Since a homomorphic image of a Hilbert ring is again a Hilbert ring, it is clear from Krull's result that if $R[\{X_i\}_{i=1}^{\infty}]=S$ is a Hilbert ring, then for each maximal ideal m of R, the field R/m must have uncountable cardinality (for $S/mS\cong(R/m)[\{X_i\}_{i=1}^{\infty}]$). In Theorem 3.3 of [2], Gilmer asserts that if R is a Hilbert ring and if $\{X_\lambda\}_{\lambda\in\Lambda}$ is an infinite set of indeterminates such that for each maximal ideal m of R, the cardinality of the field R/m is greater than that of the set Λ, then $S=R[\{X_\lambda\}]$ is a Hilbert ring. However, this assertion is incorrect as can be seen, for example, by taking R to be a 1-dimensional Noetherian domain containing an uncountable field and having a countably infinite number of maximal ideals. For R with this property, (0) is the intersection of the maximal ideals of R so that R is Hilbert. But for any maximal ideal m of R, the local ring R_m is a non-Hilbert ring that is a countably generated R-algebra, and hence a homomorphic image of the polynomial ring $R[\{X_i\}_{i=1}^{\infty}]$. Therefore $R[\{X_i\}_{i=1}^{\infty}]$ is not Hilbert. A specific example of such a ring R is the example given in [2, p. 211]. Let $\mathbf{C}$ be the field of complex numbers and let R be the localization of the polynomial ring $\mathbf{C}[X]$ at the multiplicative system generated by $\{X-\alpha\mid\alpha\in\mathbf{C}\setminus\mathbf{Z}\}$. Contrary to what is asserted in [2] and repeated in [6, Example 174, p. 145], for this ring R the polynomial ring $R[\{X_i\}_{i=1}^{\infty}]$ is not a Hilbert ring.

Gilmer informs me that the error in the proof of Theorem 3.3 of [2] occurs on page 210, 15 lines from the bottom, where it is stated that P_σ is an ideal. The above example also shows that the sufficiency assertion in Corollary 3.4 of [2] in order that $S=R[\{X_\lambda\}]$ be Hilbert is incorrect; and since Corollary 3.4 is used in the proof of Theorem 3.5 of [2], the status of this result is in need of clarification. A ring R is Hilbert if and only if each finitely generated R-algebra that is a

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