THE RANGE OF THE RESIDUE FUNCTIONAL FOR THE CLASS S_p

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Let U denote the unit disk $\{z: |z|<1\}$. For $0<p<1$, the class S_p will consist of all functions $g(z)$ which are meromorphic and univalent in U and in addition are normalized so that $g(0)=0$, $g'(0)=1$ and $g(p)=\infty$. Define the set

$$\Omega_p=\{a: a=\operatorname{Res}_{z=p} g(z),\ g\in S_p\}.$$

In this note we prove the following:

THEOREM. $\Omega_p=\{-p^2(1-p^2)^\epsilon : |\epsilon|\leqslant 1\}$.

Proof. The proof consists of a mutual inclusion argument.

Suppose that $a\in\Omega_p$. Then $a=\operatorname{Res}_{z=p} g(z)$ for some $g(z)\in S_p$. Let S denote the class of all functions $f(z)$ which are analytic and univalent in U and are normalized so that $f(0)=0$ and $f'(0)=1$. Then a short argument shows that the function

$$f_c(z)=\frac{cg(z)}{c+g(z)} \quad (-c\notin g(U))$$

belongs to S and that $a=-f_c^2(p)/f_c'(p)$. We shall apply the Golusin Inequalities [1, p. 898] to the function $f_c(z)$. For each $f\in S$, we have

$$\left|\sum_{n=1}^{N}\sum_{k=1}^{N}\lambda_n\bar{\lambda}_k \log\left(\frac{f(z_n)-f(z_k)}{z_n-z_k}\frac{z_nz_k}{f(z_n)f(z_k)}\right)\right| \leqslant \sum_{n=1}^{N}\sum_{k=1}^{N}\lambda_n\bar{\lambda}_k \log\left(\frac{1}{1-z_n\bar{z}_k}\right),$$

where the z_n $(0<|z_n|<1)$ are distinct and the λ_n are arbitrary complex numbers. For $z_k=z_n$, the quotient is interpreted as a derivative. We apply these inequalities with $k=n=N=1$, $\lambda_1=1$ and $z_1=p$ to obtain the inequality

$$\text{(1)}\qquad \left|\log\frac{p^2f'(p)}{f^2(p)}\right|\leqslant \log\frac{1}{1-p^2}.$$

This inequality was originally discovered by Grunsky [2]. Setting $f(z)=f_c(z)$ in (1), we obtain

$$|\log(-a)-\log p^2|\leqslant \log\frac{1}{1-p^2}.$$

It follows that

$$\log(-a)=\log p^2+\epsilon\log(1-p^2)$$

where $|\epsilon|\leqslant 1$. Exponentiating and multiplying by -1, we obtain $a=-p^2(1-p^2)^\epsilon$, which was what we wanted.

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