TWO COUNTABILITY PROPERTIES OF SETS OF MEASURES

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1. Introduction. Let X be a (Hausdorff) topological space and let $C(X)$ denote the space of bounded continuous real-valued functions on X. The space of (non-negative) bounded σ-additive Baire measures on X is denoted by $M_\sigma(X)$ ($M_\sigma^+(X)$). This paper deals with the following countability properties:

(a) A subset M of $M_\sigma(X)$ is called *countably separated* (c.s.) if there exists a sequence $\{f_n\}$ in $C(X)$ such that for every μ and $\nu \in M$

$$(*) \qquad \int f_n \, d\mu = \int f_n \, d\nu \quad \text{for all } n \Rightarrow \mu = \nu.$$

(b) A subset M of $M_\sigma(X)$ (resp. $M_\sigma^+(X)$) is called *countably determined* (c.d.) in $M_\sigma(X)$ (resp. in $M_\sigma^+(X)$) if there exists a sequence $\{f_n\}$ in $C(X)$ such that for every $\mu \in M_\sigma(X)$ (resp. $\mu \in M_\sigma^+(X)$) and $\nu \in M$

$$\int f_n \, d\mu = \int f_n \, d\nu \quad \text{for all } n \Rightarrow \mu \in M.$$

Countability properties of this kind occur naturally in classical and functional analysis, probability theory and general topology. Here are some examples.

The classical moment problem (see VII.3 in [6]) relates to $\mathbf{R}$ and the particular sequence $f_n(x) = x^n$, $x \in \mathbf{R}$. It is clear that if μ, ν are carried by a bounded closed interval, then $(*)$ holds. If μ, ν are arbitrary Baire measures on $\mathbf{R}$, $(*)$ does not hold, even if all moments are finite (see example on page 227 in [6]). However, a different sequence $\{f_n\}$ exists such that $(*)$ holds for every μ and $\nu \in M_\sigma(\mathbf{R})$, that is, $M_\sigma(\mathbf{R})$ is c.s. In fact this is true in a more general set-up (see §4).

The c.s. property is related to the separability of $C(X)$ as follows: $M_\sigma(X)$ is c.s. if and only if $C(X)$ is separable in the weak topology $\sigma(C(X), M_\sigma(X))$, or equivalently in any locally convex topology which yields $M_\sigma(X)$ as dual space (see §4).

A topological space Y is called *separably submetrizable* [20] if there exists a sequence $\{g_n\}$ in $C(Y)$ which separates points of Y. It is clear that Y is separably submetrizable if and only if Y with its Baire σ-algebra is a countably separated measurable space [5, p. 6] if and only if the set $M = \{\delta_y : y \in Y\}$ of Dirac measures on Y is c.s.

If M is a c.s. subset of $M_\sigma(X)$ and $\{f_n\}$ is as in the definition of the c.s. property, the sequence $g_n : M \to \mathbf{R}$, $n = 1, 2, \ldots$, with

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