ON THE MEAN BOUNDARY BEHAVIOR AND THE TAYLOR COEFFICIENTS OF AN INFINITE BLASCHKE PRODUCT

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Introduction. If $\{z_n\}$ is a sequence (finite or infinite) of complex numbers of modulus less than 1 such that $\Sigma(1-|z_n|)<\infty$, then the Blaschke product

$$B(z)=\prod_n \frac{|z_n|}{z_n}\frac{z_n-z}{1-\bar{z}_n z}$$

converges uniformly on compact subsets of the unit disc U. We let $\mathcal{B}_\infty$ denote the set of Blaschke products whose zero sequences are infinite. In this paper we show that

$$\inf_{B\in\mathcal{B}_\infty}\overline{\lim_{r\to 1}}(1-r)^{-1}\int_{-\pi}^{\pi}(1-|B(re^{i\theta})|)^2\frac{d\theta}{2\pi}=\max_{0\leqslant x<1}(1+\sqrt{1-x})\,{}_2F_1({}^{1/2}{}_2{}^{1/2};x)=\gamma_0$$

where ${}_2F_1({}^{1/2}{}_2{}^{1/2};x)$ is a hypergeometric function. Using one of Gauss' identities for the hypergeometric functions and tables for the complete elliptic integrals (see [2, pp. 608–609]), we obtain the estimate $\gamma_0\geqslant .285$.

We have two applications of this result. In [2], Newman and Shapiro showed that if $B(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}_\infty$ then $\overline{\lim}_{n\to\infty}n|a_n|\geqslant 1/\pi=.3183\cdots$. We improve this to show that if $B(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}_\infty$ then $\overline{\lim}_{n\to\infty}n|a_n|\geqslant\sqrt{\gamma_0/2}\geqslant .37749$. In the other direction we modify a method of Newman and Shapiro [2] to show that if $\epsilon>0$ is given, there is a $B(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}_\infty$ such that $\overline{\lim}\,n|a_n|\leqslant 2/e+\epsilon=.735\cdots+\epsilon$. We conjecture that if $B(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}_\infty$ then $\overline{\lim}_{n\to\infty}n|a_n|\geqslant 2/e$.

It is well known that if $B(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}_\infty$ then $\sum_{n=1}^{\infty}n|a_n|^2=\infty$. As another application of our main theorem we improve this to show that if $B(z)=\sum_{n=0}^{\infty}a_n z^n\in\mathcal{B}_\infty$ then

$$\overline{\lim_{n\to\infty}}\sum_{k\in I_n}k|a_k|^2\geqslant\frac{\gamma_0}{8}\geqslant .0356,$$

where $I_n=\{k: k \text{ is an integer}, 2^n\leqslant k<2^{n+1}\}$.

We wish to thank Dick Askey for identifying a certain series that arose in our calculations as a hypergeometric function, and for showing us how to use Gauss' formula and the tables to evaluate it.

1. We begin with a lemma. For $0<r<1$ define $\varphi_r(z)=(z-r)/(1-rz)$.

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