THE EULER–LAGRANGE EQUATIONS FOR EXTREMAL HOLOMORPHIC MAPPINGS OF THE UNIT DISK

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1. Introduction. Extremal problems for holomorphic mappings of the unit disk were considered by various authors ([3], [5], [6]), but only for the class of univalent functions. For a class of mappings to a domain $D \subset \mathbf{C}^n$ these problems were studied only when D is the unit disk. One reason for this limitation, I believe, was an absence of really interesting functionals. However, some years ago, Royden [4] introduced the functional $\|f'(0)\|$ for mappings of the unit disk to a domain $D \subset \mathbf{C}^n$. The supremum of this functional gives us the infinitesimal norm for the Kobayashi metric at a point $z = f(0)$. But extremal mappings in this case are much more interesting because they are invariant under biholomorphic transformations and, hence, are connected with some invariants. In particular, their boundary values may coincide with so called Moser chains [1]. But for a proof of the last conjecture we should know, at least, that boundary values lie on the boundary of the domain. We prove here this property for large classes of functionals and domains.

The standard tool for the study and computation of extremals in the calculus of variations are the Euler–Lagrange equations. In our paper, we deduce them in the case of pseudoconvex domains. In the last section we show how these equations help to find extremals for some types of domains. The author hopes that further studies in the complex calculus of variations will give us a clearer understanding of biholomorphic invariants.

Similar results were proved by different methods by Lempert [2] for Royden's functional and strongly linear convex domains of class C^∞.

2. Notations and preliminary results. Let $\Delta_r = \{\zeta \in \mathbf{C} : |\zeta| < r\}$ be the disk of radius r on the complex plane and $\Delta = \Delta_1$. As usual we shall denote by H or H^p the spaces of all holomorphic functions or of those whose boundary values lie in L^p. We define H_n, H_n^p, L_n^p as a direct sum of n copies of H, H^p, L^p. If $D \subset \mathbf{C}^n$, then $H(\Delta, D)$ is the set of all holomorphic mappings of Δ to D. We denote by A the subspace of H, consisting of functions continuous up to the boundary; A_n means a direct sum of n copies of A.

In addition, we shall use the following notations: $S_r = \partial \Delta_r$, $S = S_1$; if $f = (f_1, \ldots, f_r)$, $h = (h_1, \ldots, h_n)$, then $(f, h) = \Sigma f_j h_j$, $|f| = \Sigma |f_j|$; $\rho(A, B)$ is the distance between sets A and B; $\bar{A}$ means the closure of A and if u is a function, then

$$\Delta u = \left(\frac{\partial u}{\partial z_1}, \frac{\partial u}{\partial z_2}, \ldots, \frac{\partial u}{\partial z_n} \right).$$

Received January 20, 1983
Michigan Math. J. 30 (1983).