SURFACES IN MINKOWSKI 3-SPACE ON WHICH H AND K ARE LINEARLY RELATED

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1. Introduction. In this paper, we study surfaces in Minkowski 3-space M^3 on which mean curvature H and extrinsic curvature K satisfy a non-trivial linear relation $\alpha+\beta H+\gamma K\equiv 0$. Most results are based on formalisms developed in [3], which extend to the case of indefinite metric complex analytic techniques one might have expected to apply only in the Riemannian case.

On spacelike or timelike surfaces in M^3 with $\alpha+\beta H+\gamma K\equiv 0$ and $\beta^2\neq 4\alpha\gamma$, we show the existence of a certain holomorphic quadratic differential associated with the geometry of the immersion. This allows the introduction of special coordinates, and identifies three different flat metrics, among them the exotic metric $\Gamma=\alpha\mathrm{I}+\beta\mathrm{II}+\gamma\mathrm{III}$ studied by J. A. Wolf in [10]. That Γ is flat on similar surfaces in Euclidean 3-space E^3 was observed by Darboux in [2], a fact we learned recently from Wolf. The use of flat metrics here yields some information in-the-large about the surfaces in question.

There is a rich variety of surfaces in M^3 on which H or K is constant. (See [1], [4], [6] and [9] for examples.) Moreover, H and K are linearly related on any surface equidistant in M^3 from a surface on which H or K is constant. We show below that a spacelike or timelike surface in M^3 on which $\alpha+\beta H+\gamma K\equiv 0$ with $\beta^2\neq 4\alpha\gamma$ is equidistant from at least one surface with H or K constant. In addition, we extend to M^3 the classical theorem of Bonnet (see [3]) which associates to a surface of constant $H\neq 0$ (resp. $K>0$), an equidistant surface of constant $K>0$ (resp. $H\neq 0$). This extension is known to geometers, but seems not to be in the literature.

We assume C^∞ smoothness wherever possible. The symbols α,β,γ and c always denote constants.

2. Formal preliminaries. Suppose that S is an oriented surface, and that $A=E\,dx^2+2F\,dx\,dy+G\,dy^2$ and $B=L\,dx^2+2M\,dx\,dy+N\,dy^2$ are real quadratic forms with $\det A\neq 0$. Compute the curvatures $H=H(A,B)$, $K=K(A,B)$ and $H'=H'(A,B)$ by setting

$$2H=\mathrm{tr}_A B,\qquad K=\det B/\det A,\qquad 2H'=\sqrt{H^2-K}$$

with $iH'<0$ in case $H^2<K$. Denote the intrinsic curvature of A by $K(A)$. Wherever $H'\neq 0$, define the skew forms $A'=A'(A,B)$ and $B'=B'(A,B)$ by

$$H'A'=B-HA,\qquad H'B'=HB-KA.$$

Anywhere on S, the form $W=W(A,B)$ is given by

$$\sqrt{|\det A|}\,W=\begin{vmatrix} dy^2 & -dx\,dy & dx^2\\ E & F & G\\ L & M & N\end{vmatrix}.$$

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