A LATTICE POINT PROBLEM IN HYPERBOLIC SPACE

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1. Introduction. We consider the model of n-dimensional hyperbolic space which is given by the interior of the unit ball, $B=\{x: |x|<1\}$, where $x=(x_1,x_2,\dots,x_n)\in\mathbf{R}^n$ and $|x|=(\Sigma x_i^2)^{1/2}$. Lines in the space are arcs of circles orthogonal to the unit sphere, $S=\{x: |x|=1\}$, and angle is Euclidean angle. The hyperbolic metric ρ is derived from the differential

$$d\rho=\frac{2|dx|}{1-|x|^2}$$

and the hyperbolic lines are geodesics for this metric.

A Moebius transform preserving B is a product of an even number of inversions in spheres orthogonal to S and such transforms preserve the hyperbolic metric ρ. We denote by M the full group of all Moebius transforms preserving B. If G is a discrete subgroup of M and we select a point $x\in B$, then the collection of G-equivalents of x form a lattice of points in B. We shall be concerned in this paper with the way in which such a lattice is distributed in B.

Suppose x_1,x_2 are two points of B and s is a positive real number. For the discrete group G we define the counting function $N(s,x_1,x_2)$ to be the number of transforms $\gamma\in G$ such that $\rho(x_1,\gamma(x_2))<s$. We are concerned with the asymptotic behavior of $N(s,x_1,x_2)$ as s approaches infinity - this can be viewed as the hyperbolic analog of the Gauss circle problem.

The Dirichlet region D for the group G is defined by

$$D=\{x\in B: \rho(x,0)<\rho(\gamma(x),0) \text{ all } \gamma\in G, \gamma\neq I\}.$$

Now if γ is a Moebius transformation we denote by $\gamma'(x)$ the Jacobian matrix of γ at x and by $|\gamma'(x)|$ the positive number such that $\gamma'(x)/|\gamma'(x)|$ is orthogonal. In other words $|\gamma'(x)|$ is the linear change of scale at x, the same in all directions. Since $|\gamma'(x)|=(1-|\gamma(x)|^2)(1-|x|^2)^{-1}$ (see [1: ch. II]) then $|\gamma'(x)|<1$ if and only if $|\gamma(x)|>|x|$ and we see that

$$D=\{x\in B: |\gamma'(x)|<1 \text{ all } \gamma\in G, \gamma\neq I\}.$$

Hyperbolic volume V in B is derived from the differential

$$dV=\frac{2^n dx_1\,dx_2\dots dx_n}{(1-|x|^2)^n},$$

where $x=(x_1,\dots,x_n)$. We denote by $V(G)$ the hyperbolic volume of D. In this paper we are concerned solely with the situation when $V(G)<\infty$ (the infinite volume case is discussed in an earlier paper of the author [11]).

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