A REMARK ON QUASI-CONFORMAL MAPPINGS AND BMO-FUNCTIONS

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Let $G \subset \mathbf{R}^n$ $(n \geqslant 2)$ be a domain and let $u: G \to \mathbf{R}$ be a locally integrable function. We say that u has *bounded mean oscillation in G,* and denote $u \in \mathrm{BMO}(G)$, if

$$\|u\|_{*,G} \equiv \sup_{B \subset G} \left[\frac{1}{m(B)} \int_B |u(x) - u_B| \, dx \right] < \infty.$$

Here the supremum is taken over all balls $B \subset G$; $m(B)$ stands for the Lebesgue-measure of B and u_B for the mean value of u over B, i.e.

$$u_B = \frac{1}{m(B)} \int_B u(x) \, dx.$$

H. M. Reimann [5] has established a close connection between quasi-conformal mappings and the spaces $\mathrm{BMO}(G)$ by proving the following theorems:

1. THEOREM ([5: Theorem 4]; see also [4: p. 58]). *If $f: G \to G'$ is a K-quasi-conformal homeomorphism, then*

$$\frac{1}{C} \|u\|_{*,G'} \leqslant \|u \circ f\|_{*,G} \leqslant C \|u\|_{*,G'} \tag{1}$$

for all functions $u \in \mathrm{BMO}(G')$. The constant C in (1) *depends only on K and the dimension n.*

2. THEOREM ([5: Theorem 3]). *If an orientation preserving homeomorphism $f: G \to G'$ has the properties*
   (a) *f is differentiable* a.e. *and $f \in$* ACL,
   (b) *the mapping $u \to u \circ f$ is a bijective isomorphism of the spaces* $\mathrm{BMO}(G')$ *and* $\mathrm{BMO}(G)$ *for which $\|u \circ f\|_{*,G} \leqslant C \|u\|_{*,G'}$,*

then f is quasi-conformal.

For definitions of quasi-conformal and ACL mappings see [8].

The purpose of this note is to show that by localizing the problem the analytic assumptions (a) in Theorem 2 can be dropped. More precisely, we shall prove

3. THEOREM. *Let $f: G \to G'$ be an orientation preserving homeomorphism. If there exists a constant C such that*

$$\frac{1}{C} \|u\|_{*,D'} \leqslant \|u \circ f\|_{*,D} \leqslant C \|u\|_{*,D'} \tag{2}$$

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