THE ELEMENTARY THEORY OF NORMAL FROBENIUS FIELDS

Moshe Jarden

Introduction. The Galois stratification is introduced by Fried and Sacerdote [3] in order to establish an explicit primitive recursive decision procedure for the elementary theory of finite fields. This method is further developed in [2] and leads to a primitive recursive decision method for Frobenius fields. In order to be more explicit we consider a given Hilbertian field K with an elimination theory, in the sense of [2], and let M be a Frobenius field that contains K. If G is a profinite group, then we denote by $\mathrm{Im}\, G$ the set of all finite quotient groups of G. In particular, if we denote by $G(M)$ the absolute Galois group of M, then $\mathrm{Im}\, G(M)$ is the set of all finite groups that can be realized over M. It is proved in [2] that if $\mathrm{Im}\, G(M)$ is a primitive recursive set of groups, then the Galois stratification method leads to a primitive recursive decision procedure for the theory of perfect Frobenius fields M' that contain K and satisfy $\mathrm{Im}\, G(M) = \mathrm{Im}\, G(M')$.

This result is generalized in Section 1 of this work. We consider a class Π of profinite groups each of which appears as the absolute Galois group of a Frobenius field M. This class is supposed to be equipped with a primitive recursive algorithm to determine for given $m+n$ finite groups $G_1, \ldots, G_m, H_1, \ldots, H_n$ whether or not there exists a $P \in \Pi$ such that $G_1, \ldots, G_m \in \mathrm{Im}\, P$ and $H_1, \ldots, H_n \notin \mathrm{Im}\, P$. We denote by $\mathfrak{M}$ the class of all perfect Frobenius fields M that contain K and satisfy $G(M) \in \Pi$, and show how to modify the arguments in [2] in order to establish a primitive recursive procedure for the theory of $\mathfrak{M}$.

This procedure is applied in Section 2 to the set Π of all normal closed subgroups of the free profinite group $\hat{F}_\omega$ on $\aleph_0$ generators. It follows from the results of Melnikov [11] that each of these groups is indeed isomorphic to an absolute Galois group of a perfect Frobenius field. These fields are therefore called *normal Frobenius fields*. Moreover, Melnikov's characterization of these groups leads to an explicit algorithm for Π as in the preceding paragraph. It follows that the theory of normal Frobenius fields that contain K is primitive recursive via Galois stratification.

The decidability of the theory of all perfect Frobenius fields that contain K is hereby reduced to the above group theoretic decision problem for the class Π of all absolute Galois groups of Frobenius fields. An affirmative solution to this problem has been recently given by Haran and Lubotzky [6].

1. Galois stratification for a class of strongly projective groups. A profinite group P is said to be *projective* if for every epimorphism $\alpha: G \to H$ of profinite groups and every homomorphism $\gamma: P \to H$ there exists a homomorphism

---

Received June 1, 1982.

Partially supported by the fund for basic research administered by the Israel Academy of Sciences and Humanities.

Michigan Math. J. 30 (1983).