STABLE BASE LOCI OF REPRESENTATIONS OF ALGEBRAIC GROUPS

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Introduction. It is well known that the ring of invariant functions under the action of a group G on an affine k-algebra R need not again be an affine k-algebra. A counterexample to the original problem posed by Hilbert (the 14th problem) was first given by Nagata [7] and was suggested by earlier work of Zariski [12] and Rees [8]. This paper investigates the connections between the ideas discussed in [7] and [12] from the point of view of quotient spaces. Zariski's reformulation of the original 14th problem showed that the matter rests with the behavior of certain linear systems on "almost canonical" projective varieties associated to the pair R and G. We describe these linear systems here as the base loci of the canonical rational maps determined by invariant functions of a given degree (cf. Section 2). The stable behavior of these linear systems plays a key role in the problem of finite generation (Proposition 2.2).

The rational maps determined by these linear systems are regular on certain open sets and on suitable domains, called quotient domains, actually determine an orbit map. The existence of a sufficiently large quotient domain also plays a role in our main result (4.4) which asserts that stable base loci (cf. Section 2) and sufficiently large quotient domains give finite generation. We give interpretations of these results in the case of Nagata's counterexample in Examples 2.1 and 5.3.

We now fix our terminology. All schemes will be reduced algebraic k-schemes, with k a fixed algebraically closed field. A variety is a separated integral scheme. Almost all schemes appearing after Section 1 will be varieties. All algebraic groups are assumed to be affine algebraic varieties. For any irreducible scheme X we identify $\Gamma(X, O_X)$ with the subring of everywhere defined rational functions in $k(X)$ - the function field of X. Unless otherwise stated, "points" will mean closed points.

1. Generalities on group actions and linear systems. This section gives a brief summary of the results on actions of algebraic groups on varieties and the theory of linear systems which will be used in the following sections. They are given here essentially for convenience of reference.

1.1. Let G be an algebraic group acting rationally on a scheme X. A pair (Y, q) consisting of a scheme Y and a morphism $q: X \to Y$ is a geometric quotient of X by G denoted $X \bmod G$ if the following conditions hold:

(i) q is open and surjective

(ii) $q_*(O_X)^G = O_Y$

(iii) q is an orbit map; i.e., the fibers of closed points are orbits.

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