ON PRIME SEQUENCES OVER AN IDEAL

L. J. Ratliff, Jr.

1. Introduction. In [10], Rees introduced the concepts of an asymptotic sequence, an asymptotic sequence over an ideal, and a prime sequence over an ideal. In [4] it was shown that most of the basic results known to hold for R-sequences have a valid analogue for asymptotic sequences, and it was noted there that, conversely, sometimes a new result on R-sequences can be found by proving the R-sequence version of a new result on asymptotic sequences. The main purpose of this paper is to illustrate this converse phenomenon and thus to derive some new information on R-sequences. Actually, in the present case, the results are the "prime sequence over an ideal" version of "asymptotic sequence over an ideal" results.

Specifically, in [6, (5.6.1) and (5.7)] it is shown that $b_1, \ldots, b_s$ are an asymptotic sequence over an ideal I in a local ring R if and only if $b_1, \ldots, b_s, u$ are an asymptotic sequence in the Rees ring $\mathcal{R} = \mathcal{R}(R, I)$, and this holds if and only if $b_1, \ldots, b_s, u$ are an asymptotic sequence in $\mathcal{R}_{\mathfrak{M}}$, where $\mathfrak{M}$ is the maximal homogeneous ideal in $\mathcal{R}$. Then, among other things, it is shown that: the I-forms of $b_1, \ldots, b_s$ are an asymptotic sequence in the form ring $\mathfrak{F}(R, I)$ [6, (5.10)]; the images of $b_1, \ldots, b_s$ in $R[I/b_1]$ are an asymptotic sequence [6, (7.4)]; each permutation of $b_1, \ldots, b_s$ is an asymptotic sequence over I [6, (6.2)]; $b_1, \ldots, b_s$ are an asymptotic sequence in R [6, (6.4)]; and, any two maximal asymptotic sequences over I have the same length [2]. The prime sequence over an ideal version of each of these results is proved in §2, and certain additional results are also proved, such as: the form ring result just mentioned actually characterizes a prime sequence over an ideal and

$$(b_1^{e_1}, \ldots, b_s^{e_s})R \cap (I + B_s)^n = \sum_{i=1}^{s} b_i^{e_i}(I + B_s)^{n - e_i}$$

for all positive integers e_i and for all $n \geqslant 0$, where $B_s = (b_1, \ldots, b_s)R$.

Since most of the results mentioned in the preceding paragraph are rather natural to consider for any type of sequence of elements, the R-sequence versions probably would have been found without knowing that the asymptotic sequence versions are true. But even so, the close analogy between the two versions of the results does nicely illustrate the converse phenomenon mentioned above.

Prime sequences over an ideal seem to have some interesting and useful properties. Hopefully the results in this paper will be useful in any future research on such elements.

Received September 10, 1982.

Research on this paper was supported in part by the National Science Foundation, Grant MCS8001597-A02.

Michigan Math. J. 30 (1983).