ITERATES OF HOLOMORPHIC SELF-MAPS OF THE UNIT BALL IN $\mathbf{C}^N$

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Introduction. The sequence of iterates of a holomorphic map of the unit disc $\mathbf{D}$ into itself with no fixed points in $\mathbf{D}$ was studied by J. Wolff [7] and A. Denjoy [1]. They showed that for such a function the iterates converge, uniformly on compact subsets of $\mathbf{D}$, to a unimodular constant. In Section 1 of this paper we consider the generalization of this question to holomorphic, fixed point free self-maps of the unit ball in $\mathbf{C}^N$. We will show that in this case also the sequence of iterates converges, uniformly on compact subsets of the ball, to a constant of norm 1. The basic tool we use is a theorem of W. Rudin [4] which characterizes the fixed point set of a holomorphic map of the ball into itself as an affine subset of the ball.

The one variable Denjoy–Wolff theorem is often stated to include holomorphic self-maps of the disc which fix one point in the disc, but which are not conformal automorphisms of the disc. In this case the entire sequence of iterates still converges to a constant, the interior fixed point. In Section 2 we consider the iteration of maps with fixed points in the ball in higher dimensions.

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1. Maps with no interior fixed points. Let B, or B_N if we wish to indicate the dimension explicitly, be the open unit ball in $\mathbf{C}^N$, in the Euclidean metric. Denote by $H(B;B)$ the family of all holomorphic maps of B into itself. For $f \in H(B;B)$ we denote the iterates of f by f_n:

$$f_1 = f, \qquad f_{n+1} = f \circ f_n \qquad n = 1, 2, 3, \ldots$$

Since $H(B;B)$ is a normal family, every sequence of iterates of f contains a subsequence which converges, uniformly on compact subsets of B. We will examine the possible subsequential limits of $\{f_n\}$ according to the fixed point character of f. Note that a subsequential limit of iterates of $f \in H(B;B)$ need not belong to $H(B;B)$. However the following lemma shows that this can only happen if the limit is a constant map of norm 1.

LEMMA 1.1. *Let $F: B \to \bar{B}$ be holomorphic. Then either $F(B) \subseteq B$ or $F(z) \equiv \zeta$ in ∂B, for all z in B.*

Proof. Suppose there is a z_0 in B with $F(z_0) = \zeta \in \partial B$. Set $G(z) = (1 + \langle z, \zeta \rangle)/2$, so G belongs to $A(B)$, the algebra of functions holomorphic in B and continuous

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