ON ZEROS OF p-ADIC FORMS

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1. Introduction. In the 1930's E. Artin conjectured (see [3, p. x]) that a form F of degree d in n variables with coefficients in a p-adic field Q_p must have a nontrivial zero in that field if $n > d^2$. He was aware that for each d and each p there is a form of degree d in d^2 variables with coefficients in Q_p with no nontrivial p-adic zero; e.g., the reduced norm of a central simple division algebra over Q_p. As a first step towards Artin's conjecture, R. Brauer [5] showed that there is a function $\phi_p(d)$ such that if $n > \phi_p(d)$, then F has a nontrivial p-adic zero. Terjanian [16] disproved Artin's conjecture by exhibiting a 2-adic quartic form in 18 variables with no nontrivial 2-adic zero; later [17] he gave such an example with 20 variables. Generalizing Terjanian's construction, Browkin [6] gave counterexamples for each prime p, but always in fewer than d^3 variables. Recently Arhipov and Karačuba [1, 2] greatly improved on this by showing that for each p there are infinitely many d such that

$$\phi_p(d) > \exp\left(\frac{d}{(\log d)^2(\log\log d)^3}\right).$$

By introducing a more efficient principle of p-adic interpolation (Lemma 1), we sharpen their result slightly.

THEOREM 1. *Let p be a given prime and suppose $\epsilon > 0$. For infinitely many d there is a form F in $\mathbf{Z}[x_1, \dots, x_n]$ of degree d with*

$$n > \exp\left(\frac{d}{(\log d)(\log\log d)^{1+\epsilon}}\right)$$

such that if $a_1, \dots, a_n \in \mathbf{Z}$ and $F(a_1, \dots, a_n) \equiv 0 \pmod{p^d}$, then $a_1 \equiv \cdots \equiv a_n \equiv 0 \pmod{p}$.

It is not clear how close to best possible the above might be. The upper bound for $\phi_p(d)$ that one obtains from Brauer's argument is an iterated exponential which is very much larger than the lower bound we have obtained.

It would be nice to know precisely when $\phi_p(d) = d^2$. Meyer [14] found that $\phi_p(2) = 4$ for all p. Demyanov [10] and Lewis [13] independently showed that $\phi_p(3) = 9$ for all p (for other proofs see Springer [15] and Davenport [9]). Ax and Kochen [4] and Ersov [11, 12] independently proved there exists a function $p_0(d)$ such that $\phi_p(d) = d^2$ for all $p > p_0(d)$. Cohen [8] demonstrated that it is possible, at least in principle, to compute an upper bound for $p_0(d)$. It is interesting to note that in all the known examples for which $\phi_p(d) > d^2$ one has d even, composite and divisible by $p-1$. Thus it could be that these are the only

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