SOME REMARKS ON NIELSEN EXTENSIONS
OF RIEMANN SURFACES

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Let S be a Riemann surface of finite type $(g; n, m)$, that is, a surface of genus g with n punctures and m holes. Assume that $6g - 6 + 2n + 3m > 0$ and $m > 0$. Then S can be represented as U/G where U denotes the upper half-plane and G a torsion-free Fuchsian group of the second kind. Let $\mathfrak{I}$ be the set of maximal open intervals on $\mathbf{R} \cup \{\infty\}$ on which G acts discontinuously. For each $I \in \mathfrak{I}$ the stabilizer of I in G is generated by a hyperbolic element whose axis $A(I)$ is the non-Euclidean line joining the endpoints of the interval. The Nielsen convex region $N(G)$ is the complement of the union of the closures of all the half-planes bounded by I and $A(I)$, $I \in \mathfrak{I}$. The surface $S_0 = N(G)/G$ is the Nielsen kernel of S; S is the *Nielsen extension* of S_0. Every surface has a Nielsen extension [1].

Given a surface S_0 let S_k be the Nielsen extension of S_{k-1} or equivalently let S_{k-1} be the Nielsen kernel of S_k, for $k \in \mathbf{Z}$. Define the *infinite Nielsen extension* of S_0 to be $S_\infty = S_1 \cup S_2 \cup \cdots$ and define the *infinite Nielsen kernel* of S_0 to be $S_{-\infty} = S_{-1} \cap S_{-2} \cap \cdots$.

The purpose of this paper is to obtain several inequalities concerning the lengths of certain geodesics of Nielsen extensions and kernels. The Poincaré metric is used throughout and the length of a boundary curve denotes the Poincaré length of the geodesic to which it is freely homotopic.

THEOREM 1. *Let S be a finite Riemann surface with a boundary curve of length l. Then the length of the corresponding boundary curve of the Nielsen kernel of S is greater than $2l$.*

Proof. Assume that $S = U/G$ has a boundary curve C corresponding to $X: z \to e^l z$. Let f be the conformal map that takes the Nielsen convex region of S onto U and which fixes 0, 1, and ∞. Then the Nielsen kernel of S is $S_{-1} = U/fGf^{-1}$. The function $f_1 : z \to z^2$ maps the first quadrant, which contains the Nielsen convex region of S, onto U. Then $f = f_2 \circ f_1$ for some conformal map f_2 which takes a region $U_1 \subsetneq U$ onto U.

Let C_0 be the corresponding boundary curve of S_{-1} and let l_0 be its length. Let d denote length in the Poincaré metric of U. Since U_1 is a proper subset of U, $d(C_0)$ is greater than $d(f_2^{-1}(C_0))$. C_0 is a simple curve in U whose endpoints are identified by fXf^{-1}. No other points of C_0 are equivalent under any other element of the group fGf^{-1}. Therefore the only points of $f_2^{-1}(C_0)$ equivalent under any element of $f_1 G f_1^{-1}$ are its endpoints, which are identified by $f_1 X f_1^{-1}$. So $d(f_2^{-1}(C_0))$ cannot be less than the length of the shortest geodesic joining the semicircles $x^2 + y^2 = 1$ and $x^2 + y^2 = e^{2l}$, $y > 0$. Therefore

$$l_0 = d(C_0) > d(f_2^{-1}(C_0)) \geqslant d(i, e^{2l}i) = 2l. \qquad \square$$