ON CERTAIN CLASSES OF ALMOST PRODUCT STRUCTURES

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1. Introduction. A. M. Naveira [2] gave a classification of Riemannian almost product structures (M, g, P) attending to the invariances of ∇P under the action of $O(p) \times O(q)$. The essential conditions defining the classes are F (foliation), C_1 (Vidal's), C_2 (minimal), C_3 (umbilical). O. Gil-Medrano [1] gave an interpretation of C_i under the general assumption of integrability.

We first show the transversal nature of the conditions C_i when integrability is assumed. Then, we give a geometric interpretation of these conditions without integrability by expressing them in terms of Lie derivatives.

Condition C_2 turns out to depend only on the volume form induced by g on the distribution $\mathcal{H}$. It can be rephrased in terms of the *expansion* of $\mathcal{H}$, which in certain sense is dual to the divergence of the complementary distribution $\mathcal{V}$, and becomes the *complementary form* of Vaisman [5] when $\mathcal{V}$ is integrable.

We see that C_3 can be written as C_1 at each point by a conformal transformation, and give an example. If in addition $\mathcal{V}$ is integrable, we have a conformal foliation.

If $\mathcal{V}$ is a conformal foliation of codimension $q \geqslant 3$, S. Nishikawa and H. Sato [3] have proved that $\text{Pont}^k(\mathcal{H}; \mathbf{R}) = 0$ in cohomology for $k > q$, by using Cartan connections and classifying spaces. In a forthcoming paper on the conformal curvature of a conformal foliation we shall give a differential geometric proof of that result for arbitrary q. Another proof with standard techniques, less conceptual but more direct, could be given from Proposition 5.1.

2. General set-up. Let (M, g, P) be a Riemannian almost-product structure, i.e. g is a Riemannian metric on M and P is an $(1,1)$ tensor field such that $P^2 = 1$, $g(P, P) = g$. Let $\mathcal{V}$ and $\mathcal{H}$ be the *vertical* and *horizontal* distributions, corresponding to the projectors $v = \frac{1}{2}(I+P)$, $h = \frac{1}{2}(I-P)$, and assume $\dim \mathcal{V} = p$, $\dim \mathcal{H} = q \neq 0$. The capitals $A, B, C, \ldots; X, Y, Z, \ldots; Q, S, T, \ldots$ will denote vector fields that are, respectively, vertical, horizontal and unrestricted. All objects are supposed C^∞.

Let ∇ be the Levi-Civita connection and put $\alpha(Q, S, T) = g((\nabla_Q P)S, T)$. Then

$$\alpha(Q, S, T) = \alpha(Q, T, S) = -\alpha(Q, PS, PT). \tag{1}$$

Let $\{e_u\}$ $(u: p+1, \ldots, p+q)$ denote in the sequel an orthonormal local base of horizontal vector fields. Then the 1-form λ is globally well defined through the local expression $\lambda(Q) = (1/q) \sum_u \alpha(e_u, e_u, Q)$, and it is clear from (1) that $\lambda = \lambda v$.

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