A MEAN VALUE THEOREM FOR ZETA FUNCTIONS ASSOCIATED WITH POSITIVE DEFINITE INTEGRAL FORMS

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1. Let $x=(x_1,x_2,\dots,x_n)\in\mathbf{R}^n$ and let $F(x)$ be a positive definite integral form on $\mathbf{R}^n$ of degree d. Let $\zeta(F,s)=\sum_{\gamma\in\mathbf{R}^n-\{0\}}F(\gamma)^{-s}$ be a zeta function associated with F. The authors have previously shown [4] that $\zeta(F,s)$ can be continued analytically in the entire complex plane except for a simple pole at $s=n/d$ and have shown that if we write $s=\sigma+it$ then

$$\zeta(F,s)\ll\frac{|t|^{n-\sigma d}}{(n-\sigma d)(n-1-\sigma d)},\quad \frac{n-1}{d}<\sigma<\frac{n}{d}. \tag{1.1}$$

In this paper the authors exhibit the following mean value theorem for $\zeta(F,s)$.

Theorem.

(1.2)

$$\int_1^T|\zeta(F,\sigma+it)|^2\,dt\ll\begin{cases}\alpha^2T^{1+(4/[d+2])(n-\sigma d)} & \text{if } \dfrac{n-(d+2)/2d}{d}\leqslant\sigma<\dfrac{n}{d}\\[2ex] T^{2(n-\sigma d)} & \text{if } \dfrac{n-1}{d}\leqslant\sigma<\dfrac{n-(d+2)/2d}{d}\end{cases}$$

where $\alpha=\min(1/|n-\sigma d|,\log T)$.

If F is quadratic, then better estimates are available; however (1.2) is an improvement for $d>2$. The authors believe that further improvements are possible and conjecture that

$$\int_1^T|\zeta(F,\sigma+it)|^2\,dt\sim T\sum a_{F(\gamma)}F(\gamma)^{-2\sigma}$$

for $\sigma>(n-\frac{1}{2})/d$ where $a_{F(\gamma)}$ is the number of solutions to $F(\gamma')=F(\gamma)$.

2. We shall need the following analogue of integration via polar coordinates.

For $x\in\mathbf{R}^n$, let $\|x\|=\max_{1\leqslant i\leqslant n}|x_i|$. Let $B=\{x\in\mathbf{R}^n\colon\|x\|=1\}$. Then each $x\in\mathbf{R}^n-\{0\}$ may be written uniquely as $x=ru$, where $r=\|x\|$ and $u\in B$.

We define a measure w on B as follows. Suppose $A\subset B$ is a Borel set. Let $\tilde{A}=\{ru\colon 0<r\leqslant 1,\ u\in A\}$. We define $w(A)=nm(\tilde{A})$, where m is Lebesgue measure on $\mathbf{R}^n$. It can be shown that if f is integrable on a set $X=\{x\in\mathbf{R}^n\colon r_1\leqslant\|x\|\leqslant r_2\}$, then

$$\int_X f\,dx=\int_B\int_{r_1\leqslant r\leqslant r_2}r^{n-1}f(ru)\,dr\,du \tag{2.1}$$

where for brevity we write du rather than $dw(u)$.

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