IDEALS OF INJECTIVE DIMENSION 1

Eben Matlis

Introduction. Throughout this paper R is an integral domain with quotient field $Q \neq R$, and $K = Q/R$. The completion of R in the R-topology is denoted by H. Let I be a non-zero ideal of R, $S = \{1 - a \mid a \in I\}$, and $E(R/I)$ the injective envelope of R/I. If $\mathfrak{J}(R)$ denotes the Jacobson radical of R, then $I \subset \mathfrak{J}(R)$ if and only if $R = R_S$. It has been proved elsewhere that $H/IH \simeq R/I$.

The main purpose of this paper is to examine the relationship between the injective envelope of R/I and the torsion-free cover of R/I in order to shed light on the condition: $\operatorname{inj.dim}_R I = 1$. This suggests consideration of the successively weaker conditions: (a) $E(R/I) = Q/I$, (b) $\operatorname{inj.dim}_R I = 1$ (i.e. Q/I is injective); and (c) $E(R/I) \subset Q/I$.

Condition (a) naturally leads to the study of the condition: (d) $Q/I \subset E(R/I)$; (i.e., Q/I is an essential extension of R/I) and the characterization of this condition is the key to the whole question. It is proved that $Q/I \subset E(R/I)$ if and only if $I \subset \mathfrak{J}(R)$ and the only ideals of R mapping onto R/I are the principal ideals of R. Another important tool in the investigation is the notion of a complemented extension A of R. Of great importance here is the proposition that if A is a complemented extension of R, and if I is the contraction of an ideal of A contained in $\mathfrak{J}(A)$, then $A = R_S$.

The main results of this paper are summarized in the following theorem.

MAIN THEOREM. (I) *The following statements are equivalent:*

(1) $E(R/I) = Q/I$.

(2) $\operatorname{Inj.dim}_R I = 1$ *and* $I \subset \mathfrak{J}(R)$.

(3) *The canonical map:* $H \to R/I$ *is a torsion-free cover.*

(II) *The following statements are equivalent:*

(1) $\operatorname{Inj.dim}_R I = 1$ (*i.e.* Q/I *is injective*).

(2) R_S *is a complemented extension of* R; $\operatorname{inj.dim}_{R_S} I_S = 1$; *and* $\operatorname{inj.dim}_{R'_S} R'_S \leqslant 1$, *where* $R'_S = \bigcap R_N\{N \in \operatorname{max\,spec} R \mid I \not\subset N\}$ *is the complement of* R_S.

(3) *The canonical map:* $H \to R/rI$ *is a torsion-free lifting for all non-zero* $r \in R$.

(III) *The following statements are equivalent:*

(1) $E(R/I) \subset Q/I$.

(2) R_S *is a complemented extension of* R *and* $\operatorname{inj.dim}_{R_S} I_S = 1$.

(3) *The canonical map* $H \to R/I$ *is a torsion-free lifting.*

In Section 1 complemented extensions of R are discussed. In Section 2 conditions (a), (b), (c), and (d) are related to the notion of complemented extensions of R. In Section 3 torsion-free covers and liftings are discussed and are related to conditions (a), (b) and (c). Finally, in Section 4 the results of the first three sections are applied to valuation rings, Noetherian domains, and h-local domains. There are examples given illustrating the first three sections, and counter-examples to possible conjectures.

Received January 19, 1982. Revision received April 29, 1982.
Michigan Math. J. 29 (1982).