MEASURES ON THE TORUS WHICH ARE REAL PARTS OF HOLOMORPHIC FUNCTIONS

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We will say that a measure μ on the torus $\mathbf{T}^2$ is the real part of a holomorphic function if the p,qth Fourier coefficient

$$\hat{\mu}(p,q) = \int_{\mathbf{T}^2} x^p y^q \, d\mu(x,y) \tag{1}$$

vanishes whenever $pq<0$. The set $\mathcal{A}$ of probability measures on $\mathbf{T}^2$ which are real parts of homomorphic functions is weak*—compact and convex. In [3] Rudin asked for a description of the extreme points of $\mathcal{A}$. Rudin's question is interesting because it concerns a phenomenon which is unique to higher dimensions; the analogous problem for the circle is trivial. In this paper we will construct some examples of extreme elements of $\mathcal{A}$.

First we establish some notation and terminology. A mapping $G: F_1 \to F_2$, where F_1 and F_2 are convex sets, will be called an *isomorphism* if it is one-to-one, onto, and preserves convex combinations. Note that isomorphisms map extreme points into extreme points. If E is a convex set and F is a convex subset of E, then F will be called a *face* of E, if $u,v \in F$, whenever $(c,u,v) \in (0,1) \times E \times E$ and $cu + (1-c)v \in F$. Note that, if F is a face of E and v is an extreme point of F, then v is an extreme point of E. A good example of a weak* closed face of $\mathcal{A}$ is $\mathcal{A}(C) = \{\mu \in \mathcal{A} \mid \mu(\mathbf{T}^2 \setminus C) = 0\}$, where C is a closed subset of $\mathbf{T}^2$. We will use B to denote the disk algebra. B can be viewed as the algebra of continuous complex valued functions on the unit circle $\mathbf{T}$ which have the property that Fourier coefficients of negative index vanish, or B can be viewed as the algebra of functions which are holomorphic on the open unit disk D and continuous on $D \cup \mathbf{T}$. In this paper we will use both viewpoints. We will assume that B is equipped with the sup-norm $\|\cdots\|$. We will indicate the Poisson kernel $\operatorname{Re}(e^{it}+w)/(e^{it}-w)$ by $P_w(e^{it})$. Finally, we let Z^k indicate the function defined by $Z^k(w) = w^k$ when $k=0,1,2,\ldots$ and by $Z^k(w) = (\bar{w})^{-k}$ when $k=-1,-2,\ldots$. We recall that $P_w(e^{it}) = \sum_{k=-\infty}^{\infty} Z^k(w)e^{-ikt}$.

EXAMPLE 1. Consider an integer $n \geqslant 2$. Define $\pi_{n,1}: \mathbf{T} \to \mathbf{T}^2$ by $\pi_{n,1}(x) = (x^{-1}, x^{n-1})$. Let $F_{n,1} = \pi_{n,1}(\mathbf{T})$. Suppose $\mu \in \mathcal{A}(F_{n,1})$. Define the measure ν on $\mathbf{T}$ by $\nu(A) = \mu(\pi_{n,1}(A))$. It is easy to show that

$$\hat{\mu}(-p,q) = \hat{\nu}(p+(n-1)q). \tag{2}$$

It follows from (1) and (2) that $\hat{\nu}(k) = 0$ whenever $|k| \geqslant n$. Thus, there is a nonnegative trigonometric polynomial g of degree $\leqslant n-1$ such that

$$\int_{\mathbf{T}^2} f(x,y)\, d\mu(x,y) = (2\pi)^{-1} \int_0^{2\pi} f(e^{-it}, e^{i(n-1)t}) g(e^{it})\, dt. \tag{3}$$

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