PROJECTIVE MODULES OVER KRULL SEMIGROUP RINGS

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1. Introduction and notation. Since the remarkable proofs of Serre's conjecture by Quillen and Suslin several years ago ([8], [9]), a number of papers have appeared generalizing the results there to other coefficient rings (e.g. [2], [7]). However, pursuing the question in a different direction suggested by the work of Anderson [1], we inquire into the class of semigroups which can replace the free abelian semigroup of monomials in the polynomial ring. Our result, strangely enough, is that the essence of Horrocks' Theorem, on which Quillen's proof relies, can be reformulated for rings arising from Krull semigroups with torsion divisor class group [3]. Using this, we show that if S is such a semigroup and A is a ring such that all finitely generated $A[G]$-projectives are extended from A whenever G is a free abelian group, then every finitely generated $A[S]$-projective is extended from A. In particular, this holds if A is a Dedekind domain by the Suslin–Swan observation on the Quillen–Suslin result ([9], [10]).

All rings are commutative with unit unless otherwise indicated. If A is a ring and M is an A-module, we let $M^{\sim}$ denote the quasi-coherent sheaf over $\mathrm{Spec}(A)$ corresponding to M. Where not specified, we use [5] as our source of results and notation on algebraic geometry. If $S \to A$ is a morphism of rings, we say M is extended from S if there exists an S-module N such that $M \cong A \otimes_S N$. If A is a local ring with maximal ideal m, then $M^{\wedge}$ denotes the m-adic completion of M.

All semigroups are commutative, cancellative with unit. Furthermore, we assume our semigroups have torsion-free total quotient group; if S is a semigroup, we denote its total quotient group by $\langle S \rangle$. For the notion of a Krull semigroup and its divisor class group and essential valuations, we refer to [3]. $\mathbf{Z}$ denotes the group of integers, and we use $\bigoplus_{\alpha \in I} \mathbf{Z}x_\alpha$ for a free group with specified bases $\{x_\alpha \mid \alpha \in I\}$. If F is this group with basis, F_+ denotes the subsemigroup of elements of F such that all of the x_α have nonnegative coefficients. Other semigroups are written multiplicatively.

Where our proofs are similar to those in [6] and [8], they are briefly sketched, with more careful attention paid to the significant differences.

2. Preliminaries. We first prove some results on Krull semigroups which we will need later.

LEMMA 2.1. *Let S be a Krull semigroup with torsion divisor class group, $\{v_\alpha \mid \alpha \in I\}$ the set of essential valuations of S, and $\beta \in I$. Then there exists a $t \in S$ such that $v_\beta(t) > 0$ but $v_\alpha(t) = 0$ for all $\alpha \in I$ with $\alpha \neq \beta$.*

Proof. By the proof of Proposition 1 in [3], and Theorem 2 of the same paper, the map $\psi : \langle S \rangle \to F = \bigoplus_{\alpha \in I} \mathbf{Z}x_\alpha$ defined by $\psi(s) = \sum v_\alpha(s) x_\alpha$ satisfies $S = \psi^{-1}(F_+)$ and $\mathrm{Cl}(S) \cong F/\mathrm{im}(\psi)$, so since $\mathrm{Cl}(S)$ is torsion, $nx_\beta \in \mathrm{im}\,\psi$ for some $n > 0$. Pick $t \in \psi^{-1}(nx_\beta)$.

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