

EXOTIC SPHERES AS STATIONARY SETS OF HOMOTOPY SPHERE INVOLUTIONS

Reinhard Schultz

This short paper applies the techniques developed by K. H. Dovermann in [5] to answer affirmatively the following question: Given a smooth homotopy sphere Σ^n , is there a smooth involution on another homotopy sphere T^{n+m} ($m > 0$) with Σ as its fixed point set? One may view this as a partial converse to the P. A. Smith Theorem [3, Chapter III] in the smooth category. Here is the specific result:

THEOREM. *Let Σ^n be a homotopy sphere. Then there is a homotopy sphere T^{2n} with a smooth involution having Σ as its fixed point set. The Pontrjagin-Thom invariants are related by $p(T) = p(\Sigma)^2$ for suitable framings of Σ and T .*

A similar result for p odd has been proved by P. Löffler, who also treated the case of involutions in case n is even (compare [7] and [8]). Our proof relies on Löffler's method together with Dovermann's machinery.

Proof of Theorem. We assume $n > 2$ since the other cases are obvious. Consider a framing of Σ with zero Hopf invariant mod 2 (such exists by [1] and [6] plus the surjectivity of the J -homomorphism if $n = 1, 3, 7$). It follows that $\Sigma \times \Sigma$ with the twist involution is a \mathbf{Z}_2 -stably framed manifold in the sense of G. Segal [10]. Suppose we consider

$$U^{2n} = \Sigma^n \times \Sigma^n \#_{\mathbf{Z}_2} - S^n \times S^n,$$

where S^n has the usual (trivial) framing and the connected sum is done equivariantly along the fixed point set. Then we have an equivariant degree 1 collapsing map $f: U \rightarrow S(\mathbf{R}[\mathbf{Z}_2]^n \oplus \mathbf{R})$ (S denotes the unit sphere) which is already a homotopy equivalence along the fixed point set. By framability this is a normal map upon which surgery may be considered as in [5].

Case 1. If n is even, then the surgery obstruction is the multisignature of U in $\mathbf{R}[\mathbf{Z}_2]$ by [5, Theorem 2.1]. This is given by the ordinary signature of U and the so-called involution signature. The former vanishes by the Hirzebruch theorem and stable parallelizability. The latter is completely determined by the normal bundle of the involution's fixed point set [2, Section 6], which is trivial by construction (this is why we needed to add $-S^n \times S^n$). Therefore we can do surgery on U as desired to get T^{2n} . By construction, T and Σ^2 are framed bordant, and hence $p(T) = p(\Sigma^2) = p(\Sigma)^2$.

Case 2. If n is odd, then the surgery obstruction is given in two stages; first, there is the ordinary Kervaire invariant of the problem, and if this vanishes there is a \mathbf{Z}_2 -valued obstruction $\text{rank}_\Lambda K_n(f) \otimes \mathbf{Z}_2 \pmod 2$, where $\Lambda = \mathbf{Z}_2[\mathbf{Z}_2]$ (see [5, Theorem 4.1]). We claim both vanish in our particular situation.

Received July 9, 1980.
Michigan Math J. 29 (1982).