PROJECTIVE MODULES OVER GROUP-ALGEBRAS OF TORSION-FREE GROUPS

Jacques Lewin

To the memory of David L. Williams.

1. Introduction. Let Q be the field of rationals and let G be a torsion-free group. If every finitely generated projective QG-module is free we say that G is Q-projective-free.

Little seems to be known about the class of Q-projective-free groups. The known examples fall into three classes: $G=F$, a free group, $G=F\times C$, the direct product of a free group and an infinite cycle, and $G=A$, a free abelian group. The first two results are due to H. Bass [2] and P. M. Cohn [6]. The abelian case to Swan [11, p. 144]. This set of examples can be enlarged slightly since the class of Q-projective-free groups is closed under direct limits and, from G. Bergman's coproduct theorems [3], free products.

There are two examples of groups which are not Q-projective-free in the literature: $G=\langle x, y \mid x^2=y^3\rangle$, the group of the trefoil knot, and G/G'', its metabelian version. These examples are due to M. J. Dunwoody [8] and P. Berridge and M. J. Dunwoody [4].

We exhibit here a class of groups which are not projective-free.

THEOREM. *Let H be a group with a subgroup G such that*

a) KH is a domain when $K=Q$ and $K=Z/pZ$.

b) G has two generators and is not free.

c) G/G' is not free abelian of rank 2.

Then QH contains a two-generator nonfree projective left ideal P. However, $P\oplus QH=QH^2$.

COROLLARY 1. *If G is a one-relator, two-generator group whose relation is neither a power nor a commutator relation, then G is not projective-free.*

COROLLARY 2. *If G is a torsion-free polycyclic-by-finite group which is projective-free, then G is nilpotent.*

As another special case, we have

COROLLARY 3. *If G is a torsion-free abelian-by-finite group which is projective free, then G is abelian.*

This generalizes a result of D. Farkas and the author and answers a question of Farkas [9, question #21].

Note in particular that the group $G=\langle x, y \mid x^2=y^2\rangle$ is not projective-free. But G has a free abelian subgroup of index two, and QG is a skew Laurent polynomial

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