BOUNDDED PROJECTIONS AND THE GROWTH OF HARMONIC CONJUGATES IN THE UNIT DISC

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This paper is dedicated by the first author to the memory of the second author. David Leroy Williams died unexpectedly on March 9, 1980; he was 42 years old. He received his Ph.D. in 1967 at the University of Michigan, under the direction of the first author. A fine mathematician, a fine friend.

1. Introduction. Let \( u \) be a harmonic function in the open unit disc \( \Delta \) and as usual denote \( M_\infty(u, r) = \sup\{|u(re^{i\theta})| : -\pi < \theta \leq \pi\} \) for \( r < 1 \). If \( u \) is bounded, elementary estimates on the conjugate Poisson kernel show that the harmonic conjugate \( \bar{u} \) satisfies the growth condition \( M_\infty(\bar{u}, r) = O(\log(1/(1-r))) \). Moreover the analytic function \( \log(1/(1-z)) \), whose imaginary part is bounded in \( \Delta \), proves that this estimate is best possible, that is, \( \log(1/(1-r)) \) cannot be replaced with a function of slower growth. On the other hand, Hardy and Littlewood [4], [5], [3, p. 83] showed that if \( M_\infty(u, r) = O((1/(1-r)^\alpha)) \), \( \alpha > 0 \), then \( M_\infty(\bar{u}, r) \) satisfies the same growth condition. We fill the gap between these two results.

More precisely, for \( x \geq 0 \) let \( \psi(x) \) be a positive increasing function for which there exists \( \alpha > 0 \) such that \( \psi(x) = O(x^\alpha) \), \( x \to \infty \). Assuming some mild regularity conditions on \( \psi \), we show in Section 3 that if \( M_\infty(u, r) = O(\psi(1/(1-r))) \), then \( M_\infty(\bar{u}, r) = O(\tilde{\psi}(1/(1-r))) \) where \( \tilde{\psi}(x) = \frac{x}{2} \int_{1/2}^x t^{-1} \psi(t) \, dt \). We also show that this estimate is best possible by constructing a harmonic function \( u \) on \( \Delta \) such that \( M_\infty(u, r) = O(\psi(1/(1-r))) \) and \( M_\infty(\bar{u}, r) \geq \tilde{\psi}(1/(1-r)) \), \( r \in [0, 1) \).

To interpret these results, one needs to know certain facts about the ratio \( \tilde{\psi}/\psi \). We shall give a detailed discussion in Section 2. Here we make some brief observations. First, if \( \psi(x) \) grows like \( x^\alpha \) then so does \( \tilde{\psi}(x) \), and one obtains the Theorem of Hardy and Littlewood. However, if \( \psi \) grows more slowly than any positive power of \( x \), then, generally speaking, \( \tilde{\psi} \) grows faster than \( \psi \). For example, if \( \psi(x) = \log(x+2) \), then \( \tilde{\psi}(x) \) grows like \( (\log(x+2))^2 \). If \( \psi(x) = 1 \), then \( \tilde{\psi}(x) \) grows like \( \log x \); thus we recapture the bounded case mentioned above.

The above discussion remains valid if we replace \( M_\infty(u, r) \) by

\[
M_1(u, r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})| \, d\theta
\]

throughout. Of course, for \( M_p(u, r) = [(2\pi)^{-1} \int |u(re^{i\theta})|^p \, d\theta]^{1/p} \), \( 1 < p < \infty \), the well-known theorem of M. Riesz [3, p. 54] says that \( M_p(u, r) = O(\psi(1/(1-r))) \) implies \( M_p(\bar{u}, r) = O(\psi(1/(1-r))) \). Therefore, in this paper we shall be concerned only with the means \( M_\infty(u, r) \) and \( M_1(u, r) \). However the referee has pointed out to us that Theorem 1 remains valid for a rather general class of norms, namely for the norm in any "homogeneous Banach space" in the sense of Y. Katznelson. This is discussed briefly in Section 7 at the end of the paper, where the relevant definitions and references are given.

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