# DE GIORGI PERIMETER, LEBESGUE AREA, HAUSDORFF MEASURE

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To Lamberto Cesari, on the occasion of his 70th birthday.

## INTRODUCTION

Let $B$ be an open region in $\mathbf{R}^n$ whose boundary $C$ is a connected, orientable $(n-1)$ dimensional manifold and whose closure is $A$. For a subset $M$ of $\mathbf{R}^N$ we use $\mu_n(M)$, $H_n^{n-1}(M)$ and $P(M)$ respectively for the Lebesgue measure, the Hausdorff $(n-1)$ dimensional measure and the de Giorgi perimeter of $M$. We are interested in comparing three "measures" of the size of $C$. These are

a) the perimeter of $A$ (or of $B$),

b) the Hausdorff $(n-1)$ dimensional measure of $C$ (or some substitute for $C$ suitable for our purpose),

and

c) the Lebesgue surface area of a mapping whose image is $C$.

The conjecture is that under rather general conditions the three measures are either all finite or all infinite. The present article is a step toward resolving this problem.

We first observe that the perimeters of $A$ and $B$ need not be equal. Either one can be infinite while the other is finite. We show here that this can occur, for $n = 3$, only when the three dimensional Lebesgue measure of $C$ is positive, and that if $\mu_3(C) > 0$ then at least one of the perimeters $P(A)$, $P(B)$ is infinite. It follows that if $\mu_3(C) = 0$ then $P(A) = P(B)$, both finite or both infinite.

Regarding the Hausdorff $(n-1)$ dimensional measure of $C$, it is well known this value is generally large compared with other "measures." Suitable substitutes for $C$ do exist in the literature. In [11], Federer considered the reduced boundary, and in [17] Vol'pert considered the essential boundary. It will be shown that the essential boundary of Vol'pert has a topological formulation in the density topology [14], [15].

For Lebesgue surface area we show if the inclusion mapping $i: C \to \mathbf{R}^n$ is collared [2], and $C$ is finitely triangulable then if either $A$ or $B$ has finite perimeter the mapping $i$ has finite integral geometric stable area [8], [9], [10]. For $n = 3$, with the collared hypothesis and the assumption $\mu_3(C) = 0$, we then have the equivalence $P(A)$ is finite if and only if the Lebesgue area of $i$ is finite.

We dedicate this paper to Lamberto Cesari in deep appreciation of the profound

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