ON THE RUSSO-DYE THEOREM

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Let A be a unital C^*-algebra and $x \in A$, $\|x\| < 1$. Denote by $n(x,A)$ the least natural number n such that x is a convex combination of n unitary elements of A. The Russo-Dye theorem asserts that $n(x,A)$ is finite. Let $n(\rho;A)$ denote the least upper bound of the numbers $n(x,A)$, where $x \in A$, $\|x\| \leq \rho$, $0 < \rho < 1$. It is known that $n(2^{-1};A) \leq 4$ and it is shown (see [3]) that if A is the C^*-algebra of continuous functions on the unit disk and $f \in A$ is the identity function, then

$$(*) \qquad n(\rho f, A) \geq 2(1-\rho)^{-1}, \quad \text{for} \quad 0 < \rho < 1,$$

which shows that $\sup_{0<\rho<1} n(\rho;A)$ is infinite.

In a seminar on operator algebras at the Math. Dept. of INCREST, A. Ocneanu raised the question of whether $n(\rho;A)$ is finite for $\rho < 1$. In this paper we answer affirmatively this question, namely we prove that

$$(**) \qquad n(\rho;A) \leq 2\pi(1+\rho)(1-\rho)^{-1} + 2.$$

To do this we follow Harris' proof of the Russo-Dye theorem ([1]). We also exhibit another class of C^*-algebras for which the inequality $(*)$ holds, namely if a C^*-algebra A contains a nonunitary isometry v, then $n(\rho v, A) \geq 2(1-\rho)^{-1}$, $0 < \rho < 1$.

This shows that in certain C^*-algebras the estimate $(**)$ is best possible, in the sense that only the constant 2π may be improved.

First we recall some definitions.

Let H be a Hilbert space and $B(H)$ the space of bounded linear operators on H; consider a contraction $x \in B(H)$, $\|x\| < 1$; denote by $D_x = (1 - x^*x)^{1/2}$, $D_{x^*} = (1 - xx^*)^{1/2}$. For $\lambda \in C$, $|\lambda| < 1/\|x\|$, let

$$\theta_x(\lambda) = D_{x^*}(1 - \lambda x^*)^{-1}(\lambda - x)D_x^{-1} = -x + \sum_{n \geq 1} \lambda^n D_{x^*} x^{*n-1} D_x$$

be the characteristic function of the contraction x (see [2, Chapter VI]). Then $\theta_x(\lambda)$ is analytic for $|\lambda| < 1/\|x\|$ and it takes unitary values for $|\lambda| = 1$. Also by the Cauchy integral formula we have $-x = \theta_x(0) = \int_0^1 \theta_x(e^{2\pi i t})\,dt$.

Thus, to obtain x as a convex combination of n unitaries, with n as small as possible, we need a good estimate for the norm of $(d/d\lambda\, \theta_x)(\lambda)$. An easy computation shows that $(d/d\lambda\, \theta_x)(\lambda) = D_{x^*}(1 - \lambda x^*)^{-2} D_x$.

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