HOMOMORPHISMS OF C^* ALGEBRAS TO FINITE AW^* ALGEBRAS

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All C^* algebras and their homomorphisms are unital, and all ideals are two-sided unless otherwise qualified.

A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all x,y in R. The ring R is *stably finite* if all rings of $n \times n$ matrices with entries from R (denoted $M_n R$) are directly finite. For C^* algebras, what is known as *finiteness* ($xx^* = 1$ implies $x^*x = 1$), is equivalent to direct finiteness [16; Theorem 27].

Stably finite rings admit a Grothendieck group (K_0) which has a natural ordering, and this in turn can lead to a great deal of structural information about the ring. For C^* algebras, the study of K_0 is becoming popular, especially for AF algebras.

I would particularly like to acknowledge the aid of Joachim Cuntz in the form of letters, helping me to understand his K_0^* and connected concepts. Conversations with Kenneth Goodearl were also of considerable value, in clarifying the proof of the existence of dimension-like functions on C^* algebras (Section 1).

Let A be a C^* algebra; following [4], [5], we define a (Cuntz's) *dimension function* as a map $D: A \to [0,1]$ satisfying:

(i) $D(1) = 1$

(ii) $D(a + b) \leq D(a) + D(b)$

(ii′) $D(a + b) = D(a) + D(b)$ if $ab = ab^* = a^*b = ba = 0$

(iii) $D(ab) \leq \operatorname{Inf}\{D(a), D(b)\}$

(iv) If $\{a_n\}$ converges to a in norm, and if there exist x_n, y_n in A so that for all n, $a_n = x_n b y_n$ for some b, then $D(a) \leq D(b)$.

Consequences of these properties include the following:

(v) $D(a) = D((a^*a)^{1/2}) = D(a^*a) = D(a^*)$

(vi) $0 \leq a \leq b$ implies $D(a) \leq D(b)$

One can show that (v) and (vi) follow from (i) through (iv), essentially as in [5]; one observes (for example, for (vi)) that $0 \leq a \leq b$ implies the closure of the right ideal generated by b contains that of a. There thus exists a sequence $\{x_n\}$ in A with $\{bx_n\}$ converging to a; apply (iv).

If $D: A \to [0,1]$ satisfies (i) through (iii) (including (ii′), and is lower semicontinuous, then (iv) (and hence (v) and (vi)) follow.

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