DIFFERENTIAL SUBORDINATIONS AND UNIVALENT FUNCTIONS

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1. INTRODUCTION

Let $p(z)$ be regular in the unit disc U and let $\psi(r,s,t)$ be a complex function defined in a domain of $\mathbf{C}^3$. With some simple conditions on ψ the present authors in [6] determined a class of functions Ψ for which

(1) $|\psi(p(z),zp'(z),z^2p''(z))| < 1, \quad \text{for } z \in U \quad \Rightarrow \quad |p(z)| < 1, \quad \text{for } z \in U.$

In addition, they determined a different class of functions Ψ for which

(2) $\operatorname{Re} \psi(p(z),zp'(z),z^2p''(z)) > 0, \quad \text{for } z \in U \quad \Rightarrow \quad \operatorname{Re} p(z) > 0, \quad \text{for } z \in U.$

If Δ represents the unit disc in (1) and the right-half complex plane in (2) then both results can be written in the form

(3) $\{\psi(p(z),zp'(z),z^2p''(z)) : z \in U\} \subset \Delta \quad \Rightarrow \quad \{p(z) : z \in U\} \subset \Delta.$

Note that in both cases Δ is simply connected and has a smooth boundary. In this paper we will show that if Δ is any simply connected domain with a "nice boundary," then there is a class of functions Ψ for which (3) is true. Actually we will prove a more general result; if Ω is a domain and Δ is a simply connected domain with a "nice boundary" we will determine a class of functions Ψ for which

(4) $\{\psi(p(z),zp'(z),z^2p''(z)) : z \in U\} \subset \Omega \quad \Rightarrow \quad \{p(z) : z \in U\} \subset \Delta.$

This basic result and applications of it in the theory of differential equations are given in section 2. In section 3 we show that the result has many important applications, especially in the theory of univalent functions; it provides elegantly short proofs for some well-known results and enables us to obtain several new results.

Since many of the results in this paper can be expressed in terms of subordination, we repeat here the definition of subordination between two functions $g(z)$ and $G(z)$ regular in U. We say $g(z)$ *is subordinate to* $G(z)$, written $g(z) \prec G(z)$, if $G(z)$ is univalent, $g(0) = G(0)$ and $g(U) \subset G(U)$.

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