ORTHOGONAL PAIRINGS OF EUCLIDEAN SPACES

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I. INTRODUCTION

1.1. By $\mathbf{R}^n$ we denote n-dimensional Euclidean space. An *orthogonal pairing* is a bilinear map $\mu:\mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^p$, such that $\|\mu(x,y)\| = \|x\|\,\|y\|$. We say that μ has type $[m,n,p]$ or is an $[m,n,p]$-pairing. An orthogonal pairing is non-degenerate in the sense that $\mu(x,y) = 0$ implies either $x = 0$ or $y = 0$. In this paper we study the problems of existence and classification of $[m,n,p]$-pairings for some positive integers m, n, p.

1.2. There are many mathematical problems where orthogonal pairings appear.

a) Orthogonal pairings are the main tool for the construction of vector fields on spheres and projective spaces (see Section 2).

b) The T-algebras of rank 3 constructed by E. B. Vinberg for studying homogeneous cones are reduced to orthogonal pairings [15].

c) Any quadratic mapping from a sphere to a sphere is homotopic to the quadratic mapping constructed in a standard way from an orthogonal pairing [16].

d) Any orthogonal (and even any non-degenerate) $[m,m,p]$-pairing generates an elliptic linear system of m differential equations of first order with m unknown functions of p variables [14, p. 273].

1.3. A. Hurwitz [7] and J. Radon [13] gave a complete answer on the existence question for $[m,n,n]$-orthogonal pairing: it exists if and only if $m \le \rho(n)$, where $\rho(n)$ is the Hurwitz-Radon number equal to $2^{c(n)} + 8d(n)$, where

$$n = 2^{c(n)}16^{d(n)}n_1, \qquad 0 \le c(n) \le 3, \quad d(n) \ge 0, \quad n_1 \text{ odd}.$$

The result was proved also by Eckmann [5] with the help of representations of finite groups and by Atiyah, Bott, Shapiro [2] with the help of classification of Clifford modules. A classification (trivial) of $[m,n,n]$-orthogonal pairing is given in these papers too.

There are only partial results for the general case. Beherend [3] and H. Hopf [6] have proved in two different ways necessity of the following condition on m, n, p for existence an orthogonal (or even non-degenerate) $[m,n,p]$-pairing: $C_p^k \equiv 0 \pmod 2), p - m < k < n$. If either m or n is ≤ 8 the condition is sufficient. A lot of papers [10], [11], [4] are devoted to necessary conditions for existence and to constructions of non-degenerate pairings (V. S. Pjasetzky has some unpublished results on orthogonalization of these pairings).

1.4. The plan of the paper is the following. In Section 2 we give a necessary

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