MAPPINGS WITH DENSE DEFICIENCY SET

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1. INTRODUCTION

There are easily constructed examples of maps between compact, orientable manifolds which have deficient points; that is, there are points y in the image for which $\# f^{-1}(y) < |\deg(f)|$ ($\#$ = cardinality). If f is a d-to-1 ($d \geq 2$) covering map of the 1-sphere, then the suspension $\Sigma f: S^2 \to S^2$ has two deficient points while further suspensions $\Sigma^{q-1} f: S^q \to S^q$ yield a map whose deficient points comprise a $(q-2)$-sphere. Let Δ_f denote the set of deficient points of a map f between orientable manifolds. For maps between 1-manifolds $\Delta_f = \phi$, and it is a consequence of a result of Hopf [4] that for maps between 2-manifolds Δ_f is discrete. In dimensions $q \geq 3$, Honkapohja [2] showed that the non-deficient points are dense and, therefore, $\dim \Delta_f \leq q-1$; and Church and Timourian [1] showed that each compact subset of Δ_f has dimension at most $q-2$.

The question was posed to the author by P. T. Church whether the deficient points could be dense; the examples constructed in this paper have this property. Specifically, for each pair of integers $q \geq 3$ and $d \geq 2$ an example is constructed of a monotone map $f: S^q \to S^q$ such that $|\deg(f)| = d$, Δ_f is a $(q-3)$-dimensional dense subset, and $f^{-1}(\Delta_f)$ is a dense subset. Since each $f^{-1}(y)$ is connected, the restriction of f is a homeomorphism from $f^{-1}(\Delta_f)$ to Δ_f.

The above situation contrasts sharply with that which occurs for discrete maps, in which case $\dim \bar{\Delta}_f \leq q-2$ [1], and for light maps, in which case $\dim \bar{\Delta}_f \leq q-1$ [1].

The techniques used to produce the examples are taken from those developed in [8]. The techniques developed in the latter paper are more systematic and "controlled" that their predecessors used in [9], [10] and [6].

A map is *monotone* provided each point-inverse is compact and connected. We define $st(a,B) = a \cup \{b \in B : b \cap a \neq \emptyset\}$ and, recursively, $st^i(a,B) = st(st^{i-1}(a,B))$.

2. PRELIMINARIES

The basic approach which will be used to construct the examples is that developed in [8]. The machinery described there is more complicated than what is needed for our current purposes. In order to have a self contained description, the necessary components with proofs will be reproduced.

The barycentric subdivision of a triangulation L is denoted βL and the nth-barycentric subdivision is defined by the recursive formula $\beta^n L = \beta(\beta^{n-1} L)$. Geometric

Received March 25, 1980. Revision received September 18, 1980.

Michigan Math. J. 28 (1981).