PSEUDO-RATIONAL LOCAL RINGS AND A THEOREM OF BRIANÇON-SKODA ABOUT INTEGRAL CLOSURES OF IDEALS

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INTRODUCTION

In the notes [23] of C. T. C. Wall, one finds that Mather raised the problem of computing for each n the smallest integer k such that, for any non-unit f in the ring $\mathbf{O}_n = \mathbf{C}\{z_1,\dots,z_n\}$ (convergent complex power series in n variables), one has $f^k \in j(f)$ where

$$j(f) = (\partial f/\partial z_1,\dots,\partial f/\partial z_n)\mathbf{O}_n$$

is the jacobian ideal of f. It was known then that f always belongs to the integral closure $\overline{j(f)}$ of $j(f)$ (see Section 1 below), a fact which implies the existence for a given f of an integer k such that $f^k \in j(f)$. Shortly afterwards, Saito proved (see [20]) that if one assumes that the origin is an isolated critical point of f, then the inclusion $f \in j(f)$ holds if and only if f is a quasi-homogeneous polynomial in some coordinate system, and therefore ([15, Section 9]) the monodromy of the fibration over $\mathbf{D} - \{0\}$ (where $\mathbf{D} = \{t \in \mathbf{C} \mid |t| < \eta\}$) defined by f in a neighborhood of $0 \in \mathbf{C}^n$ is finite. More generally, in [21] J. Scherk has shown that the smallest k such that $f^k \in j(f)$ is greater than or equal to the exponent of nilpotence of this monodromy. The first problem, however, is to find a bound on k valid for *any* non-unit f. This problem was solved by Briançon and Skoda, who proved:

THEOREM (see [2] for a statement which is slightly weaker, but whose proof can be modified to give:) If a non-zero ideal I in $\mathbf{O}_n$ can be generated by d elements, then for every integer $\lambda \geq 1$ we have

$$\overline{I^{\lambda+d-1}} \subseteq I^{\lambda}$$

where "$\overline{}$" denotes "integral closure" of an ideal, cf. Section 1.

In particular since $j(f)$ is generated by n elements this gives $\overline{j(f)^n} \subseteq j(f)$ and therefore $f^n \in j(f)$.

The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite transcendental deep result of Skoda in [22]. The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

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