A NOTE ON TURÁN'S METHOD

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1. STATEMENT OF RESULTS

Let $s_\nu = \sum_{n=1}^{N} b_n z_n^\nu$, where the b_n and the z_n are complex numbers, and $\nu = 0, 1, 2, \ldots$. A question of current interest is the size of $|s_\nu|$ in the special case that $b_n > 0$ and $|z_n| = 1$ for all n. In this connection, Leenman and Tijdeman [4] have shown that

(1) $$\max_{1 \le \nu \le 2N} |s_\nu| \ge \frac{s_0}{2\sqrt{N}}.$$

On the other hand, by Dirichlet's theorem on uniform approximation there is a ν, $1 \le \nu \le 6^N$, such that $\left\| \nu \frac{\arg z_n}{2\pi} \right\| \le \frac{1}{6}$ for $1 \le n \le N$. For this ν we have $\operatorname{Re} z_n^\nu = \cos(\nu \arg z_n) \ge \cos \frac{\pi}{3} = \frac{1}{2}$, so that

(2) $$\max_{1 \le \nu \le 6^N} |s_\nu| \ge \frac{s_0}{2}.$$

It is easy to see that we may have $s_\nu = 0$ for $1 \le \nu \le N - 1$ (take $z_n = e(n/N)$, $b_n = 1$ for $1 \le n \le N$), so the range of ν in (1) is essentially as short as one may consider. Furthermore, $|s_\nu| \le s_0$ for all ν, so we cannot hope to improve on (2) by more than a constant if we consider longer ranges of ν. Thus the two estimates (1) and (2) represent the extreme situations. In what follows we obtain (1) and (2) by a unified method which gives good lower bounds for ranges $1 \le \nu \le K$ of intermediate length as well.

THEOREM 1. *Let* $s_\nu = \sum_{n=1}^{N} b_n z_n^\nu$, *where* $b_n > 0$ *and* $|z_n| = 1$ *for all* n. *For a given* r, $r = 1, 2, 3, \ldots$, *we have*

(3) $$\max_{1 \le \nu \le 2\binom{N+r-1}{r}} |s_\nu| \ge s_0 \left(2 \binom{N+r-1}{r} \right)^{-1/2r}.$$

From this we deduce

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