HARMONIC FORMS ON NONCOMPACT RIEMANNIAN AND KÄHLER MANIFOLDS

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More than thirty years ago, S. Bochner [4] proved that if a compact Riemannian manifold has nonnegative Ricci curvature (i.e., the Ricci tensor considered as a quadratic form is nonnegative) then any harmonic 1-form is parallel and if the manifold has positive Ricci curvature (i.e., the Ricci tensor is positive definite) then the only harmonic 1-form is the 0 form. The proof technique was first to calculate the Laplacian $\triangle(\langle\alpha,\alpha\rangle)$ of the inner product $\langle\alpha,\alpha\rangle$ of a harmonic 1-form α with itself. The calculation showed that when the Ricci tensor is nonnegative $\langle\alpha,\alpha\rangle$ is subharmonic (i.e., $\triangle\langle\alpha,\alpha\rangle$ is nonnegative) so that by the maximum principle $\langle\alpha,\alpha\rangle$ is constant and $\triangle(\langle\alpha,\alpha\rangle) = 0$. The computation also showed that, when the Ricci tensor is nonnegative, $\triangle(\langle\alpha,\alpha\rangle)$ is positive at a point of the manifold if either the covariant differential $D\alpha$ is nonzero at that point or α is not zero at that point and the Ricci curvature is positive there, thus that $\triangle(\langle\alpha,\alpha\rangle) \equiv 0$ implies that $D\alpha \equiv 0$ and, if the Ricci curvature is positive at one point, that $\alpha = 0$ at that point and hence $\alpha \equiv 0$.

In the intervening years, this technique has been applied in many other situations and has been particularly useful in the study of forms on complex manifolds (cf. [9] and [20] for extensive bibliography). During this period, the technique as described has frequently been used in a modified form in which the maximum principle is dispensed with by consideration of $\int_M \langle\triangle\alpha,\alpha\rangle$, which of course vanishes if α is harmonic (cf., e.g. [9; p. 85ff]). Essentially equivalent calculations are used in this approach, and for the purpose of this paper the direct information that $\langle\alpha,\alpha\rangle$ is subharmonic is needed.

In view of the importance of the technique for compact manifolds, it is natural to seek extensions of the technique to noncompact manifolds. For such extensions, restrictions on the forms to be considered must be imposed since, for example, any noncompact Riemannian manifold admits many nonzero harmonic 1-forms (an easy way to see this is to recall that on any open Riemannian manifold there are many nonconstant harmonic functions [15] and that the differential of a harmonic function is a harmonic 1-form since $\triangle$ and d commute). A natural type of restriction is to require the forms of be L^2 (or more generally L^p, $1 \leq p < \infty$), i.e. to require $\int_M \langle\alpha,\alpha\rangle$ (respectively, $\int_M \langle\alpha,\alpha\rangle^{p/2}$) to be finite. One might then hope to establish the nonexistence of L^2 or L^p harmonic forms on certain noncompact

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