RELATION MODULES FOR EXTENSIONS OF NILPOTENT GROUPS

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1. INTRODUCTION

Let G be a group and $G = \{x_1,\ldots,x_d; r_1,\ldots,r_s\}$ a finite presentation of G, i.e., $x_1, \ldots, x_d$ generates a free group F of rank $d(F)$ and $r_1, \ldots, r_s$ are elements of F such that $G \cong F/R$, where R is the normal closure of $r_1, \ldots, r_s$ in F. In many situations, it is desirable to know the minimal number of generators, $d_F(R)$, of R as a normal subgroup of F. For example, if G is the fundamental group of a closed 3-manifold, then the maximum of the numbers $d(F) - d_F(R)$ over all finite representations must be zero [3]. Now it is notoriously difficult to determine $d_F(R)$ in general. One does not even know, if in the case G is finite, whether the number, $dF - d_F(R)$, is an invariant for G. $dF - d_F(R)$ is known not to be an invariant of G if G is infinite. Dunwoody and Pietrowski [2] have shown that the trefoil knot group $= \{a,b; a^2 = b^3\}$ has a two generator presentation needing more than one relation.

Now if one has an exact sequence of groups

$$1 \to N \to C \xrightarrow{\pi} Q \to 1,$$

then $\bar{N} = N/(N,N)$ becomes a Q-module by conjugation, $q \cdot n = \overline{cnc^{-1}}$, where $\pi c = q$. If the above sequence arises from a presentation of G, then the G-module $\bar{R}$ is called a relation module for G. Notice that any generators of N as a normal subgroup of C map to generators of $\bar{N}$ as a Q-module, i.e., $d_C(N) \geqq d_Q(\bar{N})$.

For a relation module, Gruenberg [4] has shown that if G is finite, the number $d_G(\bar{R}) - d(F)$ is an invariant for G. Moreover no examples of finite groups are known where $d_F(R) > d_G(\bar{R})$.

It is the purpose of this paper to compute the number $d_G(\bar{R})$ when G is an extension, $1 \to N \to G \to Q \to 1$, of N by Q, where N and Q are finite nilpotent groups and the orders of N and Q are relatively prime. In all that follows we shall be constantly concerned with extensions where the orders of N and Q are relatively prime. We shall refer to such an extension as a relatively prime extension. Note that such an extension is automatically split although we shall not explicitly use that fact. In the course of our investigations we shall also compute $d_G(\mathfrak{g})$, the minimal number of generators of the augmentation ideal of $\mathbf{Z}G$.

In order to state the main result we need some notation. Let $\mathbf{F}_pQ$ be semisimple and M an irreducible $\mathbf{F}_pQ$-module ($\mathbf{F}_p$ = field of p elements). Let τ_M = number of occurrences of M in $\mathbf{F}_pQ$ and if A is any $\mathbf{F}_pQ$-module, let $\tau_M(A)$ = number of

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